

Open Research Online

The Open University's repository of research publications and other research outputs

Continuous Spectra For Substitution-Based Sequences

Thesis

How to cite:

Chan, Lax (2018). Continuous Spectra For Substitution-Based Sequences. PhD thesis The Open University.

For guidance on citations see [FAQs](#).

© 2017 The Author



<https://creativecommons.org/licenses/by-nc-nd/4.0/>

Version: Version of Record

Link(s) to article on publisher's website:

<http://dx.doi.org/doi:10.21954/ou.ro.0000d19d>

Copyright and Moral Rights for the articles on this site are retained by the individual authors and/or other copyright owners. For more information on Open Research Online's data [policy](#) on reuse of materials please consult the policies page.

oro.open.ac.uk

Continuous spectra for substitution-based sequences



Lax Chan

BSc (Hons), MSc (The University of Liverpool)

A THESIS SUBMITTED TO
SCHOOL OF MATHEMATICS AND STATISTICS
THE OPEN UNIVERSITY, MILTON KEYNES
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

21st September 2017

Academic advisors:

Prof. Uwe Grimm & Dr. Ian Short

Examination committee:

Dr. Mike Whittaker (External examiner)

School of Mathematics and Statistics, University of Glasgow

Prof. Phil Rippon (Internal examiner)

School of Mathematics and Statistics, The Open University, Milton
Keynes

Dr. Robert Brignall (Chair)

School of Mathematics and Statistics, The Open University, Milton
Keynes

DECLARATION

I confirm that the material contained in this thesis is the result of independent work and joint work with my supervisors, except where explicit reference is made to the work of others. None of it has been submitted for a degree or other qualification to this or any other university or institution.

Lax Chan

Continuous spectra for substitution-based sequences

This thesis is chiefly concerned with the continuous spectra of substitution-based sequences. First, motivated by a question of Lafrance, Yee and Rampersad [34], we establish a connection between the ‘*root- N* ’ property and the corresponding sequences that satisfy it having absolutely continuous spectrum. Then we use the recent advances in Bartlett [10, 11] to show that the Rudin–Shapiro-like sequence has singular continuous spectrum, hence does not satisfy the root- N property. This gives a negative answer to the question raised by the authors in [34].

Secondly, we use the connection we establish between the root- N property and absolute continuity to create more substitution-based sequences that have absolutely continuous/Lebesgue spectrum. This is done by modifying Rudin’s original construction [44]. We show that the binary sequences (± 1 sequences) from our modification also satisfy the root- N property and they are mutually locally derivable to the corresponding substitution sequences. This shows that the spectral properties of the substitution-based sequences are inherited from their binary counterpart.

Finally, we generalise our construction using Fourier matrices. This leads to extending Rudin’s construction to sequences with complex coefficients. This approach allows us to generate substitution sequences of any constant length greater than or equal to two. We show explicitly in the length 3 and 4 cases that these systems exhibit Lebesgue spectrum, employing Bartlett’s algorithm from Chapter 3 and mutual local derivability.

Keywords: Substitution sequences, Bartlett’s algorithm, absolute continuity, Lebesgue spectrum

PUBLICATIONS

Much of the content in this thesis has resulted in publications or has been submitted for publication:

- (i) The results in Chapter 4 have appeared in the Advances in Applied Mathematics [16].
- (ii) Part of the results in Chapter 5 have appeared in Journal of Physics: Conference Series [17].
- (iii) The results in Chapter 5 and Chapter 6 have been submitted for publication (arXiv:1706.05289).

ACKNOWLEDGEMENT

First and foremost, I would like to thank my first supervisor Prof. Uwe Grimm for giving me the opportunity to work with him and for his patience, guidance, support and encouragement throughout the process of the PhD. Many of the results presented in this thesis were inspired by our weekly conversations and collaborations. I am also greatly indebted to my second supervisor Dr. Ian Short for taking the role of being a co-supervisor despite no call of duty to do so. Thank you for being in all my weekly meetings and giving me great suggestions and ideas throughout the PhD. I feel very grateful to have had Uwe and Ian as my supervisors.

I am very thankful to the following list of mathematicians who share their ideas, work and for answering my queries: Fabien Durand, Samuel Petite, Jean-Paul Allouche, Alan Haynes, Franz Gähler, Michael Baake, Dan Rust, Daniel Lenz, Alan Bartlett and Valerie Berthe. Special thanks goes to Alan Haynes who suggested to me to go and work with Uwe in the first place.

My deepest gratitude goes to my examiners Prof. Phil Rippon and Dr. Mike Whittaker, whose valuable comments and suggestions greatly improve the presentation of this thesis.

I also thank everyone in the School of Mathematics and Statistics who have made my time here a pleasure. I am particularly grateful to Tracy Johns for sorting out all the logistics for me throughout my PhD. I cannot ask for a better office friend and PhD companion than Zillur Shabuz. Thank you for all our conversations and encouragement.

Last but not least, I would like to thank my family and my partner Sara. To my dad, thank you for your unconditional support and always encourage Max and I to pursue our dreams. Thank you to your beautiful writings, which

inspire me to put my thoughts and ideas into words here. To my mom, thank you for your unconditional love and support. I am particularly fond of the memories of us sitting in a park, drinking beer whilst you listening to my thoughts. Thank you and dad for always giving me invaluable advice to push myself to be a better person. To my dearest younger brother, thank you for being my best friend. Your ‘motto’ “life moves on” and the attitude of not taking life too seriously make the tough times I encountered more manageable. To my partner Sara, thank you for your love and support without which I would have quitted long ago.

To my family and Sara

CONTENTS

1	Introduction	1
1.1	Historical review	2
1.2	Objectives of this thesis	3
1.3	Structure of the thesis	4
2	Substitution dynamical systems	7
2.1	Basic notions	8
2.1.1	Word combinatorics	8
2.1.2	Symbolic dynamical systems	9
2.2	Substitutions	10
2.2.1	Properties of the hull \mathbb{X}	13
2.3	Measure-theoretic dynamical systems	15
2.4	Aperiodicity	17
2.5	Comparison between systems	18
2.5.1	Sliding block code	19
3	Bartlett's algorithm	21
3.1	q -adic expansions	22
3.2	Instructions and configurations	23
3.3	The correlation measures	26
3.4	Spectral hull \mathcal{K}	28
3.5	Spectral decomposition	31
3.5.1	Characterisation of different spectral components	31
3.6	Algorithm	34
3.7	Examples	35
4	Spectrum of a Rudin–Shapiro-like sequence	39
4.1	Some properties of $(\text{RSL}(n))_{n \geq 0}$	41
4.2	Spectrum of a Rudin–Shapiro-like sequence	43
4.3	Comparison with the Rudin–Shapiro sequence	48
5	Generalisation of Rudin's method: Real weights	51
5.1	Introduction	52

5.2	The Rudin–Shapiro sequence revisited	54
5.3	Modifying Rudin’s construction	56
6	Generalisation of Rudin’s method: Complex weights	75
6.1	Generalisation to complex weights	76
6.2	Examples	79
7	Conclusion	85
7.1	Summary of results obtained	86
7.2	Future questions	87
	Bibliography	91

1

INTRODUCTION

Contents

1.1	Historical review	2
1.2	Objectives of this thesis	3
1.3	Structure of the thesis	4

1.1 Historical review

For centuries, humans have been fascinated by the regular shape of crystals. Our understanding and knowledge of the internal structure of crystals were greatly improved due to the work of the Nobel prize laureates William H. Bragg, William L. Bragg and Max von Laue. They developed *X-ray crystallography* and showed that lattice-periodicity of atoms was the defining feature of a crystal. This became the accepted definition and model for solids with pure point diffraction.

In April 1982, Dan Shechtman from the Technion at Haifa made a fundamental discovery when performing diffraction experiments on various samples of an AlMn alloy. He noticed that a sample had pure point diffraction, but it also exhibited five-fold symmetry. Whilst pure point diffraction is a fingerprint of a crystal, the five-fold symmetry is not compatible with the definition of a crystal. He concluded that this sample must have long-range order (pure point diffraction) without being a crystal in the conventional sense. This discovery was initially regarded with scepticism from the crystallography community. It took two years for Shechtman to convince his colleagues until his findings were published in a joint paper in [46].

Since Shechtman's discovery, Ishimasa, Nissen and Fukano [33] found twelve-fold symmetry in a sample of NiCr alloy, and Bendersky [12] discovered another AlMn sample with ten-fold symmetry soon after. Kuo and his collaborators [50] completed the list of presently known non-crystallographic symmetries with the discovery of VNiSi and CrNiSi, which exhibit eight-fold symmetry. Nowadays, structures that are pure point diffractive but lack lattice symmetry are referred to as *quasicrystals*, a term coined by Levine and Steinhardt [35], and the definition of a crystal has been generalised to include such materials as aperiodic crystals.

From a mathematical point of view, the subject of tilings gained its importance when David Hilbert announced a list of 23 unsolved problems at the 1900 International Congress of Mathematicians. Among these problems, which shaped the course of 20th century mathematics, was problem 18, which concerns lattice and sphere packing in Euclidean space. One specific sub-part

of the question was whether there exists a polyhedron that tiles Euclidean 3-space, in such a way that it is not the fundamental domain of any space group. The first such tile in 3 dimensions was found by Karl Reinhardt in 1928, and then in 1935 Heinrich Heesch found a 2-dimensional example. Roger Penrose's interest in the subject of planar tilings was exhibited in [40], and he discovered the best known five-fold symmetric planar tiling [41]. This tiling, now known as the *Penrose tiling*, was further popularised by Martin Gardner in [30]. Although this tiling is highly ordered, it has no non-zero period. It was also shown later by De Bruijn [22] that this tiling has pure point spectrum, thereby establishing a connection between non-periodic/aperiodic tilings and quasicrystals. We refer the reader to [6, 4] for further information on historical development and background about the subject.

1.2

Objectives of this thesis

Aperiodic tilings such as the Penrose tilings have been used widely as toy models for studying quasicrystals and aperiodic phenomena. The information about the internal structure of these quasicrystals is encoded in the diffraction measure, which is the Fourier transform of the autocorrelation measure. The information obtained by the diffraction measure is the diffraction spectrum. We refer the reader to [6, Ch. 9] and [32] for background material on mathematical diffraction theory of aperiodic structures.

There are multiple ways of creating aperiodic patterns; for example, by the substitution method, cut and project schemes, or matching rules; see [6, Ch. 4,7]. In this thesis, our main object of investigation is aperiodic patterns in one dimension, generated by the substitution method. The prototiles in this case are turned into letters of a finite alphabet. We restrict our attention in particular to the situation when the substitution words are of equal length. This situation is referred to as a *constant-length substitution*, where the geometric picture and the symbolic picture coincide.

In the symbolic setting, there is a wealth of machinery such as symbolic dynamics, ergodic theory, and spectral theory to help us gain insight into these systems. We are particularly interested in the spectral properties of these

systems. As mentioned previously, the diffraction measure tells us information about the internal structure of the aperiodic systems. By an argument of Dworkin [26], the diffraction spectrum is related to part of the dynamical spectrum, which is the spectrum of a unitary operator acting on the space of complex-valued, square integrable functions; see [6, Appendix B]. For recent development and the current knowledge of the relationship between these different spectral characterisations, we refer the reader to [7].

Recently, Bartlett [10] generalized previous results of Queffélec [42] and provided an algorithm to compute the dynamical spectrum of constant-length substitutions. Throughout this thesis, we will use his algorithm to verify the spectral type of the substitution systems we investigate.

The pure point spectrum of constant-length substitution systems is well-understood due to work of Dekking [23], who shows that it is related to the *height* of the substitution and *coincidence conditions*. These two concepts will be revisited and discussed in depth in Chapter 3. Our focus is to look at the continuous spectrum of constant length substitutions, in particular the absolutely continuous/Lebesgue spectrum. The questions of interests are: are there ways to characterize the Lebesgue/absolutely continuous spectrum of substitution systems? If such characterizations exist, can we generate more examples with absolutely continuous/Lebesgue spectrum? It is worth noting that Frank [29] constructed sufficient conditions for an *aperiodic, constant-length* structures to have a Lebesgue/absolutely continuous spectrum in \mathbb{Z}^d . Our approach here is different; we extend and generalise previous work of Rudin [44].

1.3 Structure of the thesis

In Chapter 2, we will start by recalling notions from word combinatorics and symbolic dynamical systems; many of these materials were extracted from [28, Ch. 1]. Then we will define what a substitution system is. Equipped with a *shift-invariant probability measure*, we can invoke tools from *measure-theoretic dynamical systems* to study these substitution systems. The content regarding substitution systems and measure-theoretic dynamical systems are based on

the monograph by Baake and Grimm [6, Ch. 4]. We then move on to define the notion of *aperiodicity* in the context of substitution systems. This is a central notion underlying the whole thesis, as we are interested in *aperiodic, constant-length substitutions*. We finish the chapter by introducing ways to measure the “closeness” of two dynamical systems, i.e. *topological conjugacy*. In this section, we will introduce the notion of a *sliding block code*, which defines topological conjugacy between two symbolic systems. The materials in this section are mainly based on Lind and Marcus’s text [36].

The work we present in Chapter 3 is predominantly based on the recent advances of Bartlett [10, 11], where he generalises previous results of Queff  lec [42]. In particular, he provides an algorithm to determine the spectrum of aperiodic, constant-length substitutions in \mathbb{Z}^d . Since the substitutions of our investigation are all one-dimensional, we restrict our attention to the case when $d = 1$. Our aim in this chapter is to provide an exposition of Bartlett’s algorithm in the case when $d = 1$, and summarise the results needed for computing the spectrum of substitution systems we investigate later. We refer the reader to [10, 11] for details of proofs and the full generality of Bartlett’s results. Throughout this chapter, we will use the *Thue–Morse substitution* as an example to illustrate different concepts of the algorithm. We complete the chapter by introducing the *Rudin–Shapiro substitution* and computing its spectrum. This computation was done in [10, Ex. 4.3]. Here, we present the computation again as it will be used to compare with the substitution systems we investigate in later chapters.

Recently, Yee, Lafrance and Rampersad introduced a *Rudin–Shapiro-like sequence* [34]. They showed that this sequence shares a lot of similarities with the Rudin–Shapiro sequence, in particular concerning the expression of the sum of first N terms. At the end of the paper, the authors asked whether this similarity extends to the *root- N property* [2]. This question is the motivation behind Chapter 4. We first use Bartlett’s algorithm from the previous chapter to compute the spectrum of this new sequence. Then we prove the connection between the root- N property and the corresponding binary sequences with absolutely continuous/Lebesgue spectrum. Using this connection with the results obtained from Bartlett’s algorithm, we give a negative answer to the question the authors raised. We complete this chapter by discussing similarities and differences observed when computing spectra of the Rudin–Shapiro-like

sequence and the Rudin–Shapiro sequence.

The Rudin–Shapiro sequence is a paradigm of a substitution-based structure with absolutely continuous/Lebesgue spectrum. In [29], the author provided a systematic generalisation to higher dimensions. The main motivation of Chapter 5 is to provide an alternative characterization of substitution-based sequences having absolutely continuous/Lebesgue spectrum using the connection with the root- N property we establish in the previous chapter. We start by revisiting Rudin’s original approach [44], then we generalise and extend his results, yielding more substitution-based sequences. Throughout this chapter, we will use Bartlett’s algorithm to verify that the new substitution sequences do indeed have absolutely continuous/Lebesgue spectrum. By using the notion of *mutual local derivability*, we show that our construction of binary sequences from the root- N argument and the corresponding four-letter substitution sequences are mutually locally derivable, and hence inherit the same spectral type [49].

Chapter 6 is a natural continuation of Chapter 5. In Chapter 5, the coefficients in the recursion polynomials of our construction are real numbers, whereas in Chapter 6, we consider the case where the coefficients are complex numbers. This led us naturally to consider *Fourier matrices*; these are unitary matrices where rows and columns are orthogonal to each other. This allows us to produce aperiodic, constant-length substitutions of any length. We construct explicitly two examples, length 3 and length 4. For the length 3 example we once again use Bartlett’s algorithm to verify that it has Lebesgue spectrum. Moreover, we show that the ternary sequence and the nine-letter substitution sequence are mutually locally derivable, and hence they share the same spectral type. For the length 4 case, we verify the spectrum using the notion of mutual local derivability only as the computation using Bartlett’s algorithm involves rather large matrices.

We summarise the results we have obtained and discuss future research questions in Chapter 7.

2

SUBSTITUTION DYNAMICAL SYSTEMS

Contents

2.1	Basic notions	8
2.1.1	Word combinatorics	8
2.1.2	Symbolic dynamical systems	9
2.2	Substitutions	10
2.2.1	Properties of the hull \mathbb{X}	13
2.3	Measure-theoretic dynamical systems	15
2.4	Aperiodicity	17
2.5	Comparison between systems	18
2.5.1	Sliding block code	19

In this chapter, we consider a class of discrete dynamical systems, namely substitution dynamical systems. This subject has a wide range of applications to and interactions with other areas of mathematics, such as combinatorics on words, spectral theory, diophantine approximation, theoretical computer science and so on [28].

Our main goal is to use substitution dynamical systems as models of aperiodic phenomena in one dimension [6, Ch. 4].

Much of the content in this chapter is discussed in greater depth in [6, 28].

2.1 Basic notions ---

In this section, we will briefly review some of the basic notions of word combinatorics and symbolic dynamical systems. The notions and definitions will form the fundamentals for understanding substitution systems in later sections of this chapter.

2.1.1 Word combinatorics ---

Let \mathcal{A} be a finite set of d elements. The set \mathcal{A} is called an *alphabet*. The elements in the alphabet are called *letters*. A *finite word* is a finite string of letters and an *infinite word* is an infinite string of letters. We will denote by \mathcal{A}^* the set of all finite words. If we have two words, say $V = v_1 \cdots v_r$ and $W = w_1 \cdots w_s$, then the *concatenation* of the two words V and W is given by $VW = v_1 \cdots v_r w_1 \cdots w_s$. The *length* of the word is denoted $|\cdot|$. As an example, the length of the word W is s , and we write $|W| = s$.

A *one-sided sequence* is an infinite word on \mathcal{A} and is an element $u = (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$. A *two-sided sequence* on \mathcal{A} , also known as a *bi-infinite word*, is an element $u = (u_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. It is also denoted by $u = \dots u_{-1} | u_0 \dots$, where the vertical line is a reference point.

A word $v_1 \cdots v_r$ is said to *occur* at *position (index) m* in a sequence $u = (u_n)$, if there exists an index m such that $u_m = v_1, \dots, u_{m+r-1} = v_r$. In this case, we also say the word $v_1 \cdots v_r$ is a *factor* of the sequence u . A sequence u is *periodic* if there exists a positive integer T such that for every $n \in \mathbb{N}$, $u_{n+T} = u_n$.

2.1.2 Symbolic dynamical systems

Now, let us introduce some basic notions from symbolic dynamical systems. We endow the set \mathcal{A} with the discrete topology, and the set $\mathcal{A}^{\mathbb{Z}}$ with the product topology. The topology defined on $\mathcal{A}^{\mathbb{Z}}$ is the topology defined by the following metric:

$$\text{For } u \neq v \in \mathcal{A}^{\mathbb{Z}}, d(u, v) = 2^{-\min\{|n|, n \in \mathbb{Z}: u_n \neq v_n\}},$$

where the exponent looks at the smallest natural number n such that two bi-infinite sequences u and v differ at position n . Thus, the two sequences u and v are close to each other when they agree on a large region around the origin (sequence at index 0). In the study of symbolic dynamical system, one is interested in the behaviour when one ‘moves along’ the sequence $u = (u_n)_{n \in \mathbb{Z}}$. This can be described by the map $T : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$, called the *two-sided shift*:

$$T((u_n)_{n \in \mathbb{Z}}) = (u_{n+1})_{n \in \mathbb{Z}}. \quad (2.1)$$

When one restricts to $n \in \mathbb{N}$, one obtains the *one-sided shift*.

The *symbolic dynamical system* attached to the two-sided sequence $u = (u_n)_{n \in \mathbb{Z}}$ with values in \mathcal{A} is the system $(\mathbb{X}(u), T)$. The *orbit* is defined to be the set $\{T^i(u) : i \in \mathbb{Z}\}$ and $\mathbb{X}(u)$ is the closure of the orbit of the sequence u , under the \mathbb{Z} -action of the shift T .

For a word $W = w_0 \cdots w_r$, the *cylinder set* of W in $\mathbb{X}(u)$ is denoted $[W]$. It is the set $\{v \in \mathbb{X}(u) : v_0 = w_0, \dots, v_r = w_r\}$. $[W]$ is equal to the open ball $\{w' : d(w, w') < 2^{-n}\}$ and the closed ball $\{w' : d(w, w') \leq 2^{-n-1}\}$. The cylinder sets are therefore *clopen* (closed and open) sets and form a basis of open sets for the topology of $\mathbb{X}(u)$. Although this definition is given for a one-sided sequence, it can be extended to a two-sided sequence as well.

Last but not least, a sequence $u = (u_n)$ is called *minimal* or *repetitive* if every finite subword occurring in u is in an infinite number of positions with *bounded gaps*, that is to say if for every factor W , there exists an s such that for every n , W is a factor of $u_n \cdots u_{n+s-1}$.

2.2 Substitutions

A *substitution* S is a map $S : \mathcal{A} \rightarrow \mathcal{A}^*$. Roughly speaking, it is a map that replaces letters in the alphabet \mathcal{A} with a set of finite words which are formed by concatenating letters in \mathcal{A} . The map S naturally extends to maps defined over \mathcal{A}^* , $\mathcal{A}^{\mathbb{N}}$, $\mathcal{A}^{\mathbb{Z}}$ by concatenation. We also denote the extended map by S . We say a substitution is of *constant length* k if $S(a)$ is of length k for any $a \in \mathcal{A}$.

A substitution S is *injective* if each letter in the finite alphabet \mathcal{A} is mapped into distinct words. It is *bijective* if the map S defines a permutation of the alphabet in each column of the letter images.

Now, let us consider examples to illustrate the above concepts.

Example 2.2.1. Consider an alphabet with two elements, that is $\mathcal{A} = \{a, b\}$ and consider the substitution $S : a \mapsto ab, b \mapsto ba$. The map S is bijective as there is a permutation in each column of the letter images. To see it more explicitly, in the first column $a \mapsto a, b \mapsto b$, in the second column, $a \mapsto b, b \mapsto a$. This is known as the *Thue–Morse* substitution. If we apply S to the word ab , we have $S(ab) = S(a)S(b) = abba$. The substitution S is of constant length two, as both a and b are mapped into a length two words.

Example 2.2.2. Consider a two letter alphabet $\mathcal{A} = \{a, b\}$, and consider the substitution $S : a \mapsto ab, b \mapsto a$. The map S is injective, as the images of each letter is distinct, but it is not bijective, as both a and b are mapped into a in the first column. This is known as the *Fibonacci substitution*. It is a non-constant length substitution, as a and b are mapped to words of different length.

For any substitution system, one can associate matrices that encode information about the corresponding system. These matrices are obtained via *abelianization*. More concretely, given a finite alphabet of d elements, $\mathcal{A} = \{a_1, \dots, a_d\}$, consider the homomorphism $\alpha : \mathcal{A}^* \rightarrow \mathbb{Z}^d$, defined as follows:

$$\text{For every finite word } W \in \mathcal{A}^*, \alpha(W) = (|W|_{a_i})_{1 \leq i \leq d},$$

which counts the number of a_i 's in the word W . Then there is the map $M_S : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ which satisfies:

$$(M_S)_{ij} = (\alpha(S(a_i)))_{a_j},$$

where $(M_S)_{ij}$ is the (i, j) -th entry of the matrix M_S . And it is this map M_S that gives us the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{S} & \mathcal{A}^* \\ \alpha \downarrow & & \downarrow \alpha \\ \mathbb{Z}^d & \xrightarrow{M_S} & \mathbb{Z}^d \end{array}$$

and from the diagram, the commutative relation is given by

$$\text{For every } W \in \mathcal{A}^*, \alpha(S(W)) = M_S(\alpha(W)).$$

Definition 2.2.3. Let S be a substitution defined over an alphabet $\mathcal{A} = \{a_1, \dots, a_d\}$. The *substitution matrix* M_S of the substitution S is the $d \times d$ matrix whose entries at the (j, k) -th position are given by $|S(a_k)|_{a_j}$, which is the number of occurrences of a_j in $S(a_k)$.

Next, we introduce the notion of *primitivity*; this property can be observed directly from computing iterates of the substitutions or their corresponding substitution matrices.

Definition 2.2.4. A substitution S over the alphabet \mathcal{A} is *primitive* if for each index pair (i, j) , there exists some positive integer k such that every a_j is a subword of $S^k(a_i)$.

The above definition can easily be expressed in terms of the substitution matrix. Indeed, the substitution S is primitive if and only if there exists a positive integer k such that the k -th power of the substitution matrix M_S has positive entries. Thus we say that a substitution matrix is *primitive* if there exists an integer $k > 0$ such that the k -th power of the matrix has positive entries.

Remark 2.2.5. It is not clear how difficult it is to test a non-negative matrix for primitivity. Luckily, Wielandt [51] produced an upper bound for the power of matrix to be inspected. Given a $d \times d$ matrix M with real entries, it is primitive if it satisfies $M^{d^2-2d+2} > 0$.

Primitive matrices satisfy the *Perron–Frobenius theorem* [28, Thm. 1.2.6].

Theorem 2.1 (Perron–Frobenius theorem). *Let M be a primitive matrix. Then M admits a strictly positive eigenvalue λ_{PF} which dominates in modulus the*

other eigenvalues λ , i.e. $\lambda_{\text{PF}} > |\lambda|$. The eigenvalue λ_{PF} is a simple eigenvalue (Perron-Frobenius eigenvalue) and there exists an eigenvector (Perron-Frobenius eigenvector) with strictly positive entries associated with λ_{PF} .

Now let us introduce the notion of *legality*; this will allow us to define the notion of a fixed point in the substitution systems setting.

Definition 2.2.6. Let S be a substitution rule on a finite alphabet \mathcal{A} . A finite word is called *legal*, if it occurs as a subword of $S^k(a_i)$ for some $1 \leq i \leq n$ and some $k \in \mathbb{N}$.

Legal words have the property of mapping into legal words under iteration. To illustrate this, we look at Example 2.2.1. We define a sequence $(u^{(i)})_{i \in \mathbb{N}}$ by starting with a legal word $u^{(1)} = a$ and the word $u^{(i+1)}$ is given by $S(u^{(i)})$. For $i \geq 1$, and applying S , we have

$$a \mapsto ab \mapsto abba \mapsto abbabaab \mapsto \dots;$$

this is a sequence of words of increasing length that converges (in the local topology) to an infinite word that is fixed under the substitution S . More concretely, the convergence means that the ever-growing initial part of the words become stable under the iteration. In fact, if we adopt the notation $\bar{a} = b$ and $\bar{b} = a$. We have the recursion relation

$$u^{(i+1)} = u^{(i)}\overline{u^{(i)}},$$

for $i \geq 1$.

Definition 2.2.7. A bi-infinite word u is called a *fixed point* of a primitive substitution S if $S(u) = u$ and $u_{-1}|u_0$ is a legal two letter word of S .

Throughout this thesis, we will mainly work with bi-infinite fixed points instead of their one-sided counterparts. This is mainly for convenience instead of a difference in approach.

Now, the substitution S of Example 2.2.1 does not have a bi-infinite fixed point. However, the squared version i.e. S^2 does:

$$a|b \mapsto abba|baab \mapsto abbabaabbaababba|baababbaabbabaab \mapsto \dots \rightarrow u.$$

It is known that every primitive substitution has a power with a fixed point; see Lemma 4.3 of [6]. Once we have a bi-infinite fixed point u , we can consider the

set of $\mathbb{X}(u) := \overline{\{T^i u : i \in \mathbb{Z}\}}$, where T is the shift operator from Equation (2.1). This is also known as the *discrete* or *two-sided hull*. We then have a *topological dynamical system*, it is given by the pair $(\mathbb{X}(u), \mathbb{Z})$, with the continuous \mathbb{Z} -action of the shift as well as the continuous action of the substitution S .

We now combine the concept of a fixed point and primitive matrices to derive some statistical properties of the fixed point. The important property of primitive matrices implies the existence of *frequencies* for every factor of a fixed point of a primitive substitution.

Definition 2.2.8. Let u be a one-sided sequence. The *frequency* f_W of a factor W of u is defined as the limit (when n tends to infinity), if it exists, of the number of occurrences of the factor W in $u_0 u_1 \cdots u_{n-1}$ divided by n .

Once we have Theorem 2.1, we can apply it to the corresponding substitution matrices and this gives us information on the frequencies of letters that appear in a fixed point word, as a consequence of the following theorem [28, Thm. 1.2.7].

Theorem 2.2. *Let S be a primitive substitution. Let u be a fixed point of S . Then every factor of u has a frequency. Moreover, all the frequencies are positive. The frequencies of the letters are given by the coordinates of the positive eigenvector associated with the dominating eigenvalue, renormalized in such a way that the sum of its coordinates equals one.*

Example 2.2.9. Using Example 2.2.1, the substitution matrix M_S is given by

$$M_S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is primitive as $M > 0$, hence it satisfies Theorem 2.1. The Perron–Frobenius eigenvalue is $\lambda_{\text{PF}} = 2$ and the corresponding normalised eigenvector is $v_{\text{PF}} = \frac{1}{2}(1, 1)$. This tells us that the letters a and b are equally frequent asymptotically.

2.2.1 Properties of the hull \mathbb{X}

Let us now investigate and analyse the hull $\mathbb{X}(u)$ in more detail. As seen previously, we have constructed a bi-infinite fixed point using the squared version of the substitution given in Example 2.2.1 on the legal word $a|b$. But

this is not the only legal word that can lead to a fixed point of the substitution. In what follows, we shall introduce an equivalence relation that captures the idea of local isomorphism of words, in particular the infinite ones.

Definition 2.2.10. Given two words u and v in the same alphabet, we say that they are *locally indistinguishable* (abbreviated LI), denoted by $u \sim v$, when each finite subword of u is also a subword of v and vice versa.

Obviously, for finite words, they are LI if and only if they are the same. The concept of local indistinguishability is more relevant when it comes to analysing infinite words. The LI class for a given word $u \in \mathcal{A}^{\mathbb{Z}}$ is the equivalence class:

$$\text{LI}(u) := \{v \in \mathcal{A}^{\mathbb{Z}} : v \sim u\}.$$

This definition of the LI class also applies to one-sided sequences.

The LI class of a bi-infinite word u is closely linked to its hull. In fact, if the two bi-infinite words u and v are LI, their corresponding hulls are identical, as a consequence of the following Lemma [6, Lem. 4.2.].

Lemma 2.3. *If u is a bi-infinite word, its LI class is contained in the hull of u , and one has $\mathbb{X}(u) = \overline{\text{LI}(u)}$. In particular, $\mathbb{X}(u) = \mathbb{X}(v)$ holds for any two bi-infinite words u and v that are LI.*

In the previous section, we introduced the notion of minimality/repetitivity for a sequence $u = (u_n)$. There is an analogous notion in the substitution dynamical systems context.

Definition 2.2.11. The hull $\mathbb{X} \subset \mathcal{A}^{\mathbb{Z}}$ is called *minimal* if for all $u \in \mathbb{X}$, the shift orbit $\{T^i u : i \in \mathbb{Z}\}$ is dense in \mathbb{X} .

The following proposition [6, Prop. 4.1.] allows us to link the criterion for a hull to be minimal to conditions on the corresponding LI class.

Proposition 2.4. *Let u be a bi-infinite word in the finite alphabet \mathcal{A} . The following are equivalent:*

1. $\mathbb{X}(u)$ is minimal
2. $\text{LI}(u)$ is closed
3. $\mathbb{X}(u) = \overline{\text{LI}(u)}$.

Previously, we discussed the bi-infinite fixed point of the Thue–Morse substitution under S^2 (Example 2.2.1). There are other legal words that can lead to a Thue–Morse fixed point under the square of the substitution S^2 , for example $a|a$. The following result [6, Prop. 4.2] tells us that when we analyse two different fixed points of the same substitution, then they are the same locally in the sense that all the finite subwords of one fixed point must appear in another and vice versa.

Proposition 2.5. *Let S be a substitution rule on a finite alphabet. Then, any two bi-infinite fixed points u and v of S are locally indistinguishable. The same conclusion holds if u and v are fixed points of possibly different positive powers of S .*

As a consequence of combining Proposition 2.5 and Lemma 2.3, given a primitive substitution S , it does not really matter which fixed point we analyze; their hulls are identical, so they share the same properties. Therefore it makes sense to speak of the hull of a primitive substitution.

In the previous section, we introduced the notion of a sequence being repetitive and this notion gives us an alternative way to characterise the minimality of the hull, as a consequence of the following result [6, Prop. 4.3].

Proposition 2.6. *If u is a bi-infinite word on the finite alphabet \mathcal{A} , the hull is minimal if and only if u is repetitive.*

The following result allows us to connect the notion of repetitivity to the fixed point of a substitution [6, Lem.4.4].

Lemma 2.7. *Any bi-infinite fixed point of a primitive substitution on a finite alphabet is repetitive.*

Theorem 2.8. *Every primitive substitution on a finite alphabet possesses a unique hull. The hull consists of a single closed LI class.*

2.3

Measure-theoretic dynamical systems

We have introduced the notions of a symbolic dynamical system and a substitution dynamical system associated to an infinite sequence $u = (u_n)_{n \in \mathbb{Z}}$.

These systems belong to the larger class of dynamical system, namely, the *measure-theoretic dynamical systems*. Such systems have been intensively studied [19, 49] and it is this setting of measure-theoretic dynamical system that allows us to use tools from ergodic theory to derive more properties of the hull \mathbb{X} [6, Sect. 4.3].

We consider Borel, probability measures on the hull $\mathbb{X} \subset \mathcal{A}^{\mathbb{Z}}$; the set of all such measures is denoted $\mathbb{P}(\mathbb{X})$. To each element $u \in \mathbb{X}$, we associate a point (or Dirac) measure δ_u . It is defined as follows:

$$\delta_u(A) = \begin{cases} 1 & \text{if } u \in A \subset \mathcal{A}^{\mathbb{Z}} \\ 0 & \text{if } u \notin A. \end{cases}$$

Clearly $\delta_u \in \mathbb{P}(\mathbb{X})$. Now, taking into account the shifted version of u , i.e. $T^i u$, we consider

$$\mu_N := \frac{1}{2N+1} \sum_{i=-N}^N \delta_{T^i u},$$

which defines a sequence in $\mathbb{P}(\mathbb{X})$. By an application of the Banach–Alaoglu theorem [43, Thm. IV.21], the set $\mathbb{P}(\mathbb{X})$ is compact in the weak-* topology. Recall, a Borel measure μ is called *shift-invariant* if $\mu(T^{-1}(B)) = \mu(B)$ for all Borel sets B , where T is the shift operator of Equation (2.1). Since the set $\mathbb{P}(\mathbb{X})$ is compact, the sequence μ_N has a convergent subsequence, whose limit, μ , is a shift-invariant element of $\mathbb{P}(\mathbb{X})$.

Definition 2.3.1. A *measure-theoretic dynamical system* is a system (\mathbb{X}, T, μ, B) , where μ is a shift-invariant probability measure on the σ -algebra B of subsets of \mathbb{X} , and $T : \mathbb{X} \rightarrow \mathbb{X}$ is a measurable map.

Now, let us introduce the concept of *ergodicity*; this will allow us to state Birkhoff’s ergodic theorem in generality.

Definition 2.3.2. Let $\mathbb{X} \subset \mathcal{A}^{\mathbb{Z}}$ be a two-sided hull of a primitive substitution. An invariant probability measure μ on \mathbb{X} is called *ergodic* (under the action of the shift T) if the measure $\mu(B)$ for any invariant Borel set B is either 0 or 1.

Birkhoff’s ergodic theorem allows us to relate the orbit averages to the space averages. Here we present it in the setting for the two-sided hull [6, Thm. 4.2]

Theorem 2.9. Let $\mathbb{X} \subset \mathcal{A}^{\mathbb{Z}}$ be a two-sided hull, and let μ be a Borel probability measure on \mathbb{X} that is invariant under the shift T . If $f \in L^1(\mathbb{X}, \mu)$, the sequence

$\left(\frac{1}{n} \sum_{i=0}^{n-1} f(T^i u)\right)_{n \in \mathbb{N}}$ converges, for μ -almost every $u \in \mathbb{X}$ to a function $F \in L^1(\mathbb{X}, \mu)$ that is shift invariant and satisfies $\int_{\mathbb{X}} F d\mu = \int_{\mathbb{X}} f d\mu$.

Moreover, if μ is an ergodic probability measure, the function F is constant μ -almost everywhere, and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i u) = \int_{\mathbb{X}} f d\mu$$

holds for μ -almost all $u \in \mathbb{X}$.

The case where there exists only one shift invariant probability measure on \mathbb{X} is of particular interest in the study of a substitution dynamical system. A system (\mathbb{X}, T) is then called *uniquely ergodic* if there is only one shift invariant probability measure on \mathbb{X} . If a uniquely ergodic system is also minimal, then the system is called *strictly ergodic*.

The following theorem [6, Thm. 4.3] allows us to apply the concept of ergodicity in the context of substitution dynamical system.

Theorem 2.10. *Let S be a primitive substitution on a finite alphabet. Its hull \mathbb{X} is then strictly ergodic under the \mathbb{Z} -action of the shift.*

2.4 Aperiodicity

As mentioned at the beginning of this chapter, we are interested in using substitution systems to study aperiodic phenomena in one dimension. A substitution S is *aperiodic* if the corresponding hull $\mathbb{X}(u)$ has no shift periodic elements, meaning $T^k(u) = u$ implies $k = 0$ for all $u \in \mathbb{X}$. In this section, we state a few of the common characterisations of aperiodicity in one dimension.

The first characterisation is a criterion on the leading eigenvalues of the substitution matrix [6, Thm. 4.6].

Theorem 2.11. *Let S be a primitive substitution on a finite alphabet with substitution matrix M_S , and let u be a bi-infinite fixed point of S . If the PF eigenvalue is irrational, the sequence u is aperiodic.*

Theorem 2.11 gives us a straightforward way to verify aperiodicity. However, it is a sufficient but not necessary condition for aperiodicity. For example, if we

take the substitution S in Example 2.2.2, the PF eigenvalue is $\lambda_{\text{PF}} = \frac{1+\sqrt{5}}{2}$, so the substitution is aperiodic. But, if we take Example 2.2.1, the PF eigenvalue in this case is $v_{\text{PF}} = 2$. Theorem 2.11 does not apply in this situation, hence we will need some other ways to verify aperiodicity, for example via [6, Prop. 4.9].

Another characterisation of aperiodicity is based on a result of Pansiot [39, Lem. 1]. A *neighbourhood* of a letter α is a word $\gamma\alpha\beta$ which appears in the set of all subwords appearing in some substitution sequence $v \in \mathbb{X}(u)$.

Lemma 2.12. (*Pansiot's Lemma*) *A primitive substitution S which is injective on \mathcal{A} is aperiodic if and only if S has a letter with two distinct neighbourhoods.*

Lemma 2.12 is an efficient way of verifying aperiodicity. For example, let us look at the second iterate of S on the letter a from Example 2.2.1; we have $S^2(a) = abba$. The letter b in this case has two distinct neighbourhoods abb and bba , and since it is an injective substitution, it is aperiodic. Similarly, for Example 2.2.2, we look at the fourth iterate of S , $S^4 = abaab$; in this case, the letter a has two distinct neighbourhoods baa and aab . Since the substitution is injective, it is therefore aperiodic. Unfortunately, there is no higher dimensional analogue of Pansiot's lemma. If one would like to verify aperiodicity of substitutions in higher dimensions, where letters are replaced by rectangular blocks [10, 29], we would need the notion of *recognisability*; this is also referred to as the *unique decomposition property*, which means that one can locally identify the words that emerge from the substitution of a single letter. The following result was first proved by Mossé [37] in one dimension and extended by Solomyak [47] to self-similar \mathbb{R}^d tilings. It connects the notion of recognisability to aperiodicity. Here we state it in the context of substitution systems [10, Thm. 2.9].

Theorem 2.13. *A substitution S on \mathcal{A} is aperiodic if and only if for every $v \in \mathbb{X}(u)$, there exists a unique $k \in [0, q)$, where q is the length of the substitution and $w \in \mathbb{X}(u)$ such that $T^k(S(w)) = v$.*

2.5

Comparison between systems

Given any two dynamical systems, one would like to measure the “closeness” of the systems and *topological conjugacy* is a natural isomorphism notion for

dynamical systems. Given two dynamical systems (X, S) and (Y, T) , they are said to be *topologically conjugate* if there exists a homeomorphism $\phi : X \rightarrow Y$ such that $\phi \circ S = T \circ \phi$. They are semi-conjugate if it is only assumed to be continuous and onto. In this case, we say that (Y, T) is a *factor* of (X, S) .

2.5.1 Sliding block code

Suppose we have a bi-infinite sequence $u = \cdots u_{-1}|u_0 \cdots$ in a hull \mathbb{X} over a finite alphabet \mathcal{A} . We can transform u into a new sequence $v = \cdots v_{-1}|v_0 \cdots$ over another alphabet \mathbf{A} as follows. Fix some integers $-m \leq n$. To compute the i -th coordinate v_i of the transformed sequence, we use a function ϕ that depends on the “window” of coordinates of u from $i - m$ to $i + n$. Here $\phi : B_{m+n+1}(\mathbb{X}) \rightarrow \mathbf{A}$ is a fixed *block map* that allowed us transform $(m+n+1)$ -block in \mathbb{X} to symbols in \mathbf{A} , and so

$$v_i = \phi(u_{i-m}u_{i-m+1} \cdots u_{i+n}) = \phi(u_{[i-m, i+n]}) \quad (2.2)$$

Definition 2.5.1. Let \mathbb{X} be a two-sided hull over a finite alphabet \mathcal{A} , and let $\phi : B_{m+n+1} \rightarrow \mathbf{A}$ be a block map. Then the map $\phi : \mathbb{X} \rightarrow \mathbf{A}^{\mathbb{Z}}$ defined by $v = \phi(u)$ with v_i given by Equation (2.2) is called the *sliding block code* with *memory* m and *anticipation* n .

Given two two-sided hulls, \mathbb{X} and \mathbb{Y} . If $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ is a sliding block code and $u \in \mathbb{X}$, then computing ϕ at the shifted sequence $S_{\mathbb{X}}(u)$ gives the same result as shifting the image $\phi(u)$ using $S_{\mathbb{Y}}$, as a consequence of the following proposition [36, Prop. 1.5.7]

Proposition 2.14. *Let \mathbb{X} and \mathbb{Y} be two-sided hulls. If $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ is a sliding block code, then $\phi \circ S_{\mathbb{X}} = S_{\mathbb{Y}} \circ \phi$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{S_{\mathbb{X}}} & \mathbb{X} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{Y} & \xrightarrow{S_{\mathbb{Y}}} & \mathbb{Y} \end{array}$$

Earlier, we discussed briefly the notions of conjugacy, semi-conjugacy and factors. Now, we put those notions in the settings of a sliding block code. If

a sliding block code $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ is onto, then ϕ is called a *factor code* from \mathbb{X} onto \mathbb{Y} . A two-sided hull \mathbb{Y} is a *factor* of \mathbb{X} if there is a factor code from \mathbb{X} to \mathbb{Y} . A factor code is often referred to as the factor map in the literature. If φ is injective, then φ is called an *embedding* of \mathbb{X} to \mathbb{Y} . Sometimes, a sliding block code $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ has an *inverse*, more concretely, it means there is a block code $\psi : \mathbb{Y} \rightarrow \mathbb{X}$ such that $\psi(\phi(u)) = u$ for all $u \in \mathbb{X}$ and $\phi(\psi(v)) = v$ for all $v \in \mathbb{Y}$. In these circumstances, we can write $\psi = \phi^{-1}$, and we call ϕ *invertible*.

Definition 2.5.2. A sliding block code $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ is a *conjugacy* from \mathbb{X} to \mathbb{Y} , if it is invertible. Two bi-infinite hulls \mathbb{X} and \mathbb{Y} are conjugate if there is a *conjugacy* from \mathbb{X} to \mathbb{Y} .

Now, let us state a fundamental result [36, Thm. 6.2.9] from the shift dynamical system literature; it tells us that any conjugacy or homomorphism from one shift system to another is given exactly by a sliding block code.

Theorem 2.15. (*Curtis–Lyndon–Hedlund Theorem*) Suppose that $(\mathbb{X}, S_{\mathbb{X}})$ and $(\mathbb{Y}, S_{\mathbb{Y}})$ are shift dynamical systems, and $\phi : \mathbb{X} \rightarrow \mathbb{Y}$ is a continuous function. Then ϕ is a sliding block code if and only if it is a homomorphism.

As a result of the above theorem, for two substitution systems $(\mathbb{X}, S_{\mathbb{X}})$ and $(\mathbb{Y}, S_{\mathbb{Y}})$, a factor map from $S_{\mathbb{X}}$ to $S_{\mathbb{Y}}$ is the same as a factor code from \mathbb{X} to \mathbb{Y} , and a topological conjugacy from $S_{\mathbb{X}}$ to $S_{\mathbb{Y}}$ is the same as a conjugacy from \mathbb{X} to \mathbb{Y} .

3

BARTLETT'S ALGORITHM

Contents

3.1	<i>q</i>-adic expansions	22
3.2	Instructions and configurations	23
3.3	The correlation measures	26
3.4	Spectral hull \mathcal{K}	28
3.5	Spectral decomposition	31
3.5.1	Characterisation of different spectral components	31
3.6	Algorithm	34
3.7	Examples	35

The main aim of this chapter is to give an exposition of the recent advances of Bartlett [10]. By generalising and developing previous results of Queffélec [42], he provided an algorithm to determine the spectrum of aperiodic, constant length substitutions on \mathbb{Z}^d . Since the substitutions we consider throughout this thesis are one-dimensional, we will mainly focus on Bartlett's algorithm when $d = 1$.

The idea behind the algorithm is that given a shift-invariant, Borel probability measure μ on the hull $\mathbb{X} := \overline{\{T^i u : i \in \mathbb{Z}\}}$ as defined in Theorem 2.9, we are interested in the unitary translation operator $f \mapsto f \circ T$ acting on $L^2(\mathbb{X}, \mu)$, the space of complex-valued square integrable function on \mathbb{X} . We denote by \mathbb{T} , the one-dimensional torus. Take a pair of functions $f, g \in L^2(\mathbb{X}, \mu)$; there is a complex Borel measure $\sigma_{f,g}$ on the torus \mathbb{T} called the *spectral measure* for f, g with Fourier coefficients satisfying the following identity

$$\widehat{\sigma_{f,g}}(k) := \int_{\mathbb{T}} z^{-k} d\sigma_{f,g} = \int_{\mathbb{X}} (f \circ T^{-k}) \cdot \bar{g} d\mu. \quad (3.1)$$

We denote $\sigma_f := \sigma_{f,f}$. By the spectral theorem of unitary operators [11, Thm. 1.1.2], there is a *maximal function* $F \in L^2(\mathbb{X}, \mu)$ such that for any $g \in L^2(\mathbb{X}, \mu)$, the spectral measure of g is absolutely continuous with respect to F , and every measure λ that is absolutely continuous to σ_F is the spectral measure of some $g \in L^2(\mathbb{X}, \mu)$. We denote σ_F by $\sigma_{\max}(\mu)$, the *maximal spectral type* of (\mathbb{X}, T, μ) . Then by the Lebesgue decomposition theorem [6, Thm. 8.3], σ_{\max} can be separated into its *pure point*, *singular continuous* and *absolutely continuous* components on \mathbb{T} with respect to Lebesgue measure.

3.1 q -adic expansions

In this section, we will establish some arithmetic concepts. These concepts are simply consequences of the classical division algorithm on \mathbb{Z} . They will then be translated into the context of substitution systems. We refer to [10, Sect.2.1] for more details.

Fix an integer $q > 1$; then for every $n \in \mathbb{N}$ and each $k \in \mathbb{Z}$, the division algorithm modulo q^n is the following

$$k = \lceil k \rceil_n + \lfloor k \rfloor_n q^n.$$

Here $\lceil \cdot \rceil_n : \mathbb{Z} \rightarrow [0, q^n)$ is the *remainder* and $\lfloor \cdot \rfloor_n : \mathbb{Z} \rightarrow \mathbb{Z}$ is the *quotient*, modulo q^n . If we write

$$k_n := \lceil \lfloor k \rfloor_n \rceil_1 = \lfloor \lceil k \rceil_{n+1} \rfloor_n$$

for $k \in \mathbb{Z}$ and $n \geq 0$. This gives us a unique *digit sequence* $k_j \in [0, q)$ for $j = 0, \dots, n-1$, such that for $n \geq 0$

$$k = k_0 + k_1q + \dots + k_{n-1}q^{n-1} + \lfloor k \rfloor_n q^n$$

which is referred to as the n -th q -adic *expansion* of k , and we call k_n the n -th *digit* of k .

Sometimes, when adding together two numbers q -adically, it might give rise to a number larger than q in the n -th digit. We define the n -carry set for k ,

$$\Delta_n(k) = \{j \in [0, q^n) : \lfloor j + k \rfloor_n \neq 0\}.$$

We now briefly discuss the above notions in the context of constant length substitutions, and we assume they are of length q . For every $n \geq 0$, we tile \mathbb{Z} with the superblocks $[0, q^n)$ at each position of $q^n\mathbb{Z}$; these represent the domains of the translated superblock $T^j S^n(\gamma)$. The n -th quotient $\lfloor k \rfloor_n$ indicates the superblock in $q^n\mathbb{Z}$ containing k , and the n -th remainder $\lceil k \rceil_n$ tells us where k sits inside that superblock. $\lfloor j + k \rfloor_n$ represents the superblock of size q^n containing $j + k$, the n -carry set $\Delta_n(k)$ corresponds to the locations within a superblock which will leave that superblock when translated by k .

3.2 Instructions and configurations

Let S be a substitution of constant length q . For each $j \in [0, q)$, there is a map that sends $\gamma \in \mathcal{A}$ to $(S(\gamma))_j$. This maps a letter from the alphabet, to the letter that appeared at the j -th position of the word $S(\gamma)$. By definition, this is a map from $\mathcal{A} \rightarrow \mathcal{A}$; we call this map the j -th *instruction* of S and it is denoted R_j for $j \in [0, q)$. For $n \geq 0$ and $j \in [0, q^n)$, $R_j^{(n)}$ is the j -th instruction of S^n and it will from now on be referred to as the *generalised instructions* of S . The following proposition [10, Prop. 2.2] allows us to formulate precisely the connection between substitutions and the corresponding instructions.

Proposition 3.1. *Let S be a substitution of constant length q . For every $n \in \mathbb{N}, j \in \mathbb{Z}$ and $u \in \mathcal{A}^{\mathbb{Z}}$*

$$(S^n(u))_j = R_{j_0} R_{j_1} \cdots R_{j_{n-1}}(u_{\lfloor j \rfloor_n}),$$

where $\lfloor \cdot \rfloor_n$ is the quotient modulo q^n and $j_i \in [0, q)$ are the q -adic digits of j .

The above proposition tells us that finding the letter appearing at the j -th position of the bi-infinite sequence $S^n(u)$ is equivalent to looking at the composition of instruction maps R_{j_i} applied to the $\lfloor j \rfloor_n$ position of the bi-infinite sequence u . The intuition behind the proof is that the sequence $S(u)$ is obtained by concatenating the word $S(u_a)$ at the coordinate aq for $a \in \mathbb{Z}$. The letter at $(b + aq)$ -th position comes from the b -th letter of $S(u_a)$, so that $R_b(\alpha) = (S(\alpha))_b$ for $\alpha \in \mathcal{A}$ and $b \in [0, q)$, we have $(S(u))_{b+aq} = R_b(u_a)$. In this case, $b = \lceil b + aq \rceil_1$ and $a = \lfloor b + aq \rfloor_1$, which proves the base case of the theorem when $n = 1$. We refer the reader to [10] for full details of the proof.

Example 3.2.1. Recall the Thue–Morse substitution S from Example 2.2.1; the substitution $S : a \mapsto ab, b \mapsto ba$. The instructions maps are given by

$$R_0 : \begin{array}{l} a \mapsto a \\ b \mapsto b \end{array} \quad R_1 : \begin{array}{l} a \mapsto b \\ b \mapsto a \end{array}$$

Say we are interested in $(S^4(a))_{12}$, i.e. the twelfth letter of the fourth iterate on the letter a . Since this is a substitution of constant length 2, so $q = 2$, we are interested in the 2-adic expansion of 12. It is given by $12 = 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$. So in the context of Proposition 3.1, the 2-adic digits of 12 are $j_0 = 0, j_1 = 0, j_2 = 1$ and $j_3 = 1$ and $\lfloor 12 \rfloor_4 = 0$. By Proposition 3.1, we must consider $R_0 \circ R_0 \circ R_1 \circ R_1(u_0)$, where u_0 is the letter positioned at the index 0, which in this case is a . We observe that $R_0 \circ R_0 \circ R_1 \circ R_1(a) = a$. Now, $S^4(a) = abbabaabbaababba$, and $(S^4(a))_{12} = a$. We have therefore verified that $(S^4(a))_{12} = R_0 \circ R_0 \circ R_1 \circ R_1(a) = a$, in agreement with Proposition 3.1.

Given a substitution S of constant length q , a *configuration of instructions* on \mathcal{A} is a map $R : [0, q) \rightarrow \mathcal{A}^{\mathcal{A}}$ which assigns to every $j \in [0, q)$ an instruction $R_j : \mathcal{A} \rightarrow \mathcal{A}$. One can view the corresponding instruction maps $R_j : \mathcal{A} \rightarrow \mathcal{A}$ as substitutions of length one, the substitution matrices of R_j are called *instruction matrices*, which we will also denote by R_j . The sum of these instruction matrices

is the substitution matrix of the substitution S . The instruction matrices and substitution matrices for Example 2.2.1 are

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_S = \sum_j R_j = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We now describe a product of two constant length substitutions of length q . Given two alphabets \mathcal{A} and $\tilde{\mathcal{A}}$, denote by $\mathcal{A}\tilde{\mathcal{A}}$ their *product alphabet* comprising the pairs $\alpha\tilde{\gamma}$, where $\alpha \in \mathcal{A}, \tilde{\gamma} \in \tilde{\mathcal{A}}$.

Definition 3.2.2. Let S and \tilde{S} be two constant length substitutions of length q on \mathcal{A} and $\tilde{\mathcal{A}}$ respectively. Their *substitution product* $S \otimes \tilde{S}$ is a substitution on $\mathcal{A}\tilde{\mathcal{A}}$ with configuration $R \otimes \tilde{R}$ whose j -th instruction is

$$(R \otimes \tilde{R})_j : \mathcal{A}\tilde{\mathcal{A}} \rightarrow \mathcal{A}\tilde{\mathcal{A}} \quad \text{with} \quad (R \otimes \tilde{R})_j : \alpha\tilde{\gamma} \mapsto R_j(\alpha)\tilde{R}_j(\tilde{\gamma}),$$

obtained by concatenating the configurations R and \tilde{R} of the respective substitutions.

If $\tilde{\mathcal{A}} = \mathcal{A}$, then we write $\mathcal{A}^2 = \mathcal{A}\mathcal{A}$ and if $\tilde{S} = S$, then $S \otimes S$ is called the *bi-substitution* of S . The substitution matrix associated to the bi-substitution is called the *coincidence matrix* $C_S = M_{S \otimes S}$, after Queff  lec; see [42, Ch. 10]. We refer the reader to [10] for justification of the symbols we use here.

Remark 3.2.3. Given two matrices A and B , their *Kronecker product* is the matrix $A \otimes B$. In general, the matrix of a substitution product is given by the Kronecker product of the matrices of the substitutions. As an example, we take

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

their Kronecker product is given by

$$A \otimes B = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}.$$

Let us consider an example to illustrate the above concepts.

Example 3.2.4. Recall the Thue–Morse substitution S from Example 2.2.1. The product alphabet is $\mathcal{A}^2 = \{aa, ab, ba, bb\}$ and the bi-substitution is

$$\begin{aligned} S \otimes S : \quad & aa \mapsto R_0(a)R_0(a)R_1(a)R_1(a) = aabb \\ & ab \mapsto R_0(a)R_0(b)R_1(a)R_1(b) = abba \\ & ba \mapsto R_0(b)R_0(a)R_1(b)R_1(a) = baab \\ & bb \mapsto R_0(b)R_0(b)R_1(b)R_1(b) = bbaa \end{aligned}$$

As an example, the instructions maps for the element $aa \in \mathcal{A}^2$ will be $R_0 \otimes R_0 : aa \mapsto aa$ and $R_1 \otimes R_1 : aa \mapsto bb$. Therefore the entries for the instruction matrices are in terms of the pairs in the product alphabet. The instruction matrices for $S \otimes S$ are

$$\begin{aligned} R_0 \otimes R_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ R_1 \otimes R_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The coincidence matrix is given by

$$C_S = M_{S \otimes S} = \sum_j R_j \otimes R_j = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

3.3 The correlation measures

For $\alpha, \beta \in \mathcal{A}$, the *correlation measure* $\sigma_{\alpha\beta}$ on the space of complex Borel measures on the torus \mathbb{T} , is the spectral measure for the indicator functions $\mathbb{1}_{[\alpha]}$ and $\mathbb{1}_{[\beta]}$, where $[\alpha]$, $[\beta]$ are the cylinder sets of words starting with α and β respectively. Using Equation (3.1), we can compute the Fourier coefficients of the correlation measure $\sigma_{\alpha\beta}$ and obtain for $k \in \mathbb{Z}$

$$\widehat{\sigma_{\alpha\beta}}(k) = \int_{\mathbb{X}} (\mathbb{1}_{[\alpha]} \circ T^{-k}) \cdot \mathbb{1}_{[\beta]} d\mu = \mu(T^k[\alpha] \cap [\beta]). \quad (3.2)$$

The above relates the Fourier coefficients of the correlation measures $\sigma_{\alpha\beta}$ to the frequencies with which the two-letter patterns appear in substitution sequences. The following theorem [10, Thm. 3.4] connects the maximal spectral type to the correlation measure. We denote by ω_q the q -adic support measure for the q -adic roots of unity and denote by $*$ the convolution; see [10, Sect. 3.1].

Theorem 3.2. *If S is a substitution of constant length q , the maximal spectral type of (\mathbb{X}, T, μ)*

$$\sigma_{\max} \sim \sum_{\alpha \in \mathcal{A}} \omega_q * \sigma_{\alpha\alpha},$$

where \sim denotes equivalence, so the spectrum is determined by the correlation measures.

Definition 3.3.1. The vector valued measure $\Sigma := \sum_{\alpha\beta \in \mathcal{A}^2} \sigma_{\alpha\beta} e_{\alpha\beta}$ is the correlation vector for S , where $e_{\alpha\beta}$ is a unit vector correspond to the word $\alpha\beta$.

The following theorem [10, Thm. 3.6] gives us an efficient way of computing Fourier coefficients of the correlation measures.

Theorem 3.3. *(Fourier recursion theorem)*

Let S be an aperiodic substitution of constant length q . Then for $p \in \mathbb{N}$ and $k \in \mathbb{Z}$

$$\begin{aligned} \widehat{\Sigma}(k) &= \frac{1}{q^p} \sum_{j \in [0, q^p)} \left(R_j^{(p)} \otimes R_{j+k}^{(p)} \right) \widehat{\Sigma}(\lfloor j+k \rfloor_p) \\ &= \lim_{n \rightarrow \infty} \frac{1}{q^n} \sum_{j \in [0, q^n)} \left(R_j^{(n)} \otimes R_{j+k}^{(n)} \right) \widehat{\Sigma}(0) \end{aligned}$$

where $\lfloor j+k \rfloor_p$ is the quotient of $j+k$ modulo q^p and $R_j^{(n)}$ are the generalised instructions.

Let us now compute some Fourier coefficients of the correlation vector for the Thue–Morse substitution.

Example 3.3.2. First, we compute the Kronecker products of the instruction matrices R_0 and R_1 ; they are given by

$$R_0 \otimes R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_0 \otimes R_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$R_1 \otimes R_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_1 \otimes R_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, we apply the Fourier recursion theorem 3.3 for $k = 1$ and $p = 1$,

$$\begin{aligned} \widehat{\Sigma}(1) &= \frac{1}{2} \sum_{j \in [0,2)} (R_j \otimes R_{j+1}) \widehat{\Sigma}(\lfloor j+2 \rfloor_1) \\ &= \frac{1}{2} (R_0 \otimes R_1) \widehat{\Sigma}(0) + \frac{1}{2} (R_1 \otimes R_0) \widehat{\Sigma}(1) \end{aligned}$$

We can then solve this equation to find $\widehat{\Sigma}(1)$, as $\widehat{\Sigma}(0) = \frac{1}{2} \sum_{\gamma} e_{\gamma\gamma}$. This is a consequence of Michel's theorem [10, Thm. 1.1] and the fact that the Perron–Frobenius vector in this case is $(\frac{1}{2}, \frac{1}{2})$, so

$$\widehat{\Sigma}(0) = \frac{1}{2}(1, 0, 0, 1) \quad \text{and} \quad \widehat{\Sigma}(1) = (2\mathbf{I} - R_1 \otimes R_0)^{-1} R_0 \otimes R_1 \widehat{\Sigma}(0) = \frac{1}{6}(1, 2, 2, 1)$$

with the basis ordered lexicographically $e_{aa}, e_{ab}, e_{ba}, e_{bb}$.

If $k = 2$, we have

$$\begin{aligned} \widehat{\Sigma}(2) &= \frac{1}{2} \sum_{j \in [0,2)} (R_j \otimes R_{j+2}) \widehat{\Sigma}(\lfloor j+2 \rfloor_1) \\ &= \frac{1}{2} (R_0 \otimes R_0 + R_1 \otimes R_1) \widehat{\Sigma}(1) \\ &= \frac{1}{6}(1, 2, 2, 1). \end{aligned}$$

For $k > 2$, by the Fourier recursion theorem 3.3, we have

$$\widehat{\Sigma}(k) = \frac{1}{2} \sum_{j \in [0,2)} (R_j \otimes R_{j+k}) \widehat{\Sigma}(\lfloor j+k \rfloor).$$

We can therefore use such an expression to compute Fourier coefficients of the correlation vector for $k > 2$.

3.4 Spectral hull \mathcal{K}

We are interested in linear combinations of the correlation measures. For a vector $v \in \mathbb{C}^{\mathcal{A}^2}$, we write

$$\lambda_v := v^t \Sigma = \sum_{\alpha\beta \in \mathcal{A}^2} v_{\alpha\beta} \sigma_{\alpha\beta} \quad (3.3)$$

which defines a map taking a vector v onto the corresponding linear combination of the correlation measures. The following gives us a useful condition for guaranteeing positivity of the correlation measure and is due to Queffélec; see [42, Prop. 10.3].

Definition 3.4.1. For $v = (v_{\alpha\beta})_{\alpha\beta \in \mathcal{A}^2} \in \mathbb{C}^{\mathcal{A}^2}$, we let \mathbf{v} denote its associated matrix. We say v is *strongly semi-positive* if \mathbf{v} is positive semi-definite. We write $v \geq 0$ whenever $\mathbf{v} \geq 0$.

One can read the entries of a $\mathbb{C}^{\mathcal{A}^2}$ vector into the entries of its associated matrix along each row sequentially in order:

$$(a \quad b \quad c \quad d)^t \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The following lemma links the strong positivity of the vector $v \in \mathbb{C}^{\mathcal{A}^2}$ to the positivity of linear combinations of correlation measures [10, Lem. 3.10].

Lemma 3.4. *If $v \in \mathbb{C}^{\mathcal{A}^2}$ is a non-zero and strongly semi-positive vector, then $v^t \Sigma$ is a positive measure.*

Definition 3.4.2. The *spectral hull* $\mathcal{K}(S)$ of a constant length substitution S is the collection

$$\mathcal{K}(S) := \{v \in \mathbb{C}^{\mathcal{A}^2} : C_S^t = qv \text{ and } \mathbf{v} \geq 0 \text{ and } \mathbf{v}^t \widehat{\Sigma}(0) = 1\},$$

where C_S is the coincidence matrix and q is the length of the substitution.

Now, we discuss an alternative characterisation of the spectral hull which is much easier when it comes to computation. For every constant length substitution S , there is an index $h > 0$ and a partition of the alphabet $\mathcal{A} = E_1 \sqcup \dots \sqcup E_K \sqcup T$, where \sqcup denotes disjoint union, such that

- $S^h : E_j \rightarrow E_j^*$ for each $1 \leq j \leq K$,
- $\gamma \in T$ implies $S^h(\gamma) \notin T^*$,

where E_j^* and T^* denote the words formed by elements of these sets. We call the set $\{E_1, \dots, E_K, T\}$ the *ergodic decomposition* of S . The elements E_1, \dots, E_K are called the *ergodic classes* and T is the *transient class*.

Example 3.4.3. Consider the Thue–Morse substitution from Example 2.2.1. The bi-substitution was computed in Example 3.2.4. In this case, there are two ergodic classes $E_1 = \{aa, bb\}$ and $E_2 = \{ab, ba\}$ and no transient class T as the image of the two-letter word under bi-substitution are formed by concatenating those two letter words.

Let P_T denote the projection of transient elements onto the span of the transient coordinate vectors of $\mathbb{C}^{\mathcal{A}^2}$. For $w_1, \dots, w_J \in \mathbb{C}$, we write for the ergodic classes E_j

$$V_E := V_E(w_1, \dots, w_J) := \sum_{j=1}^J \sum_{\alpha\beta \in E_j} w_j e_{\alpha\beta} \in \mathbb{C}^{\mathcal{A}^2}.$$

The following proposition [10, Prop. 3.13] allows us to compute the spectral hull explicitly.

Proposition 3.5. *If S is a constant length substitution, then $v \in K(S)$ if and only if \mathbf{v} is self-adjoint with non-negative eigenvalues and satisfies*

$$\sum_{\alpha} v_{\alpha\alpha} u_{\alpha} = 1 \text{ and } v = V_E - (q\mathbf{I} - C_T^t)^{-1}(q\mathbf{I} - C_S^t)V_E,$$

where $C_T^t = C_S^t P_T$. In particular if $T = \emptyset$, then $v = V_E$; if S is a primitive substitution, then $v_{\alpha\alpha} = 1$ for all $\alpha \in \mathcal{A}$.

Let us now compute the spectral hull of the Thue–Morse substitution using Proposition 3.5.

Example 3.4.4. In Example 3.4.3, we computed the ergodic decomposition of the Thue–Morse substitution. The ergodic classes are given by $E_1 = \{aa, bb\}$ and $E_2 = \{ab, ba\}$. There is no transient class in this case hence $T = \emptyset$. Since the substitution is primitive, $\sum_{\alpha} v_{\alpha\alpha} u_{\alpha} = 1$ is equivalent to $w_1 = 1$. Now, we compute the eigenvalues of \mathbf{v} .

$$\det(\mathbf{v} - \lambda \mathbf{I}) = \det \begin{pmatrix} 1 - \lambda & w_2 \\ w_2 & 1 - \lambda \end{pmatrix} = (1 - \lambda + w_2)(1 - \lambda - w_2) \iff \lambda = 1 \pm w_2.$$

Since we want non-negative eigenvalues, w_2 satisfies $-1 \leq w_2 \leq 1$. We therefore have $v \in \mathcal{K}(S)$ if and only if $v = e_{aa} + e_{bb} + w_2 e_{ab} + w_2 e_{ba}$ for $w_2 \in [-1, 1]$. Therefore the spectral hull for the Thue–Morse substitution is

$$\mathcal{K}(S) = \{e_{aa} + e_{bb} + w_2(e_{ab} + e_{ba}) : -1 \leq w_2 \leq 1\}.$$

3.5 Spectral decomposition

In this section, we state the main result of Bartlett, which is discussed in greater length in [10, Sect. 3.4]. Recall from Theorem 3.2 that the maximal spectral type is related to the correlation measures. The following theorem shows that any positive linear combination of these ergodic measures gives rise to the maximal spectral type by convolution with ω_q .

Theorem 3.6. *If S is an aperiodic substitution of constant length q , then its spectrum is determined by the correlation vector Σ and the extremal point \mathcal{K}^* ,*

$$\sigma_{\max} \sim \omega_q * \sum_{w \in \mathcal{K}^*} \lambda_w$$

and the measures $\lambda_w = w^t \Sigma$ for $w \in \mathcal{K}^*$ are ergodic probability measures on the torus \mathbb{T} .

We refer the reader to [10, Sect. 5] for details of the proof. The above theorem tells us that these ergodic probability measures arise as linear combinations of correlation measures with coefficients in the spectral hull, which are computable via the Fourier recursion theorem 3.3 for Σ and Proposition 3.5 for \mathcal{K}^* .

3.5.1 Characterisation of different spectral components

As mentioned at the very beginning of this chapter, by the Lebesgue decomposition theorem, we can decompose σ_{\max} into its *pure point*, *singular continuous* and *absolutely continuous* components with respect to Lebesgue measure. Here, we would like to provide some ways of characterising these components.

For the pure point component, much of our understanding is due to work of Dekking [23]. Given a substitution S of constant length q , let u denote a fixed point of S . For every $k \geq 0$, consider $S_k = \{a \geq 1 : u_{a+k} = u_k\}$ and $g_k = \gcd S_k$, where \gcd stands for the greatest common divisor.

Definition 3.5.1. The *height* of a substitution is the number

$$h = \max\{n \geq 1, \gcd(n, q) = 1, n \text{ divides } g_0\}$$

and is denoted by $h := h(S)$.

The height has the following interesting properties [42, Def. 6.1]:

Remark 3.5.2. 1. $h = \max\{n \geq 1, \gcd(n, q) = 1, n \text{ divides } g_k\}$ for every $k \geq 0$. As a consequence, given an alphabet \mathcal{A} with $\text{Card}(\mathcal{A}) = s$, we have $1 \leq h \leq s$.

2. If $h = s$, u is periodic.

3. Consider for $j = 0, \dots, h$ the class

$$C_j = \{u_a : a \equiv j \pmod{h}\}.$$

Note that these classes form a partition of \mathcal{A} . If we identify in u the letters belonging to the same class C_j , we obtain a periodic sequence, and h is the smallest integer coprime to q with this property.

Let us consider some examples to illustrate the above concepts.

Example 3.5.3. Recall the Thue–Morse substitution from Example 2.2.1. We have $\text{Card}(\mathcal{A}) = 2$, so the height h must be $1 \leq h \leq 2$ using Property 1 from Remark 3.5.2. However, the height cannot be 2 in this case, because the letters a and b appear in both C_0 and C_1 , hence cannot form a partition of the alphabet using Property 3 of Remark 3.5.2. Hence, the height of the Thue–Morse substitution is 1.

Example 3.5.4. Consider the following substitution on the alphabet $\mathcal{A} = \{a, b, c, d, e\}$ given by

$$S : a \mapsto acbd, b \mapsto bdeb, c \mapsto ebae, d \mapsto ae bd, e \mapsto cbde,$$

and consider the fixed point generated by iterating S on a . The height in this case is 3 as $C_0 = \{a, d\}$, $C_1 = \{c, e\}$ and $C_2 = \{b\}$.

We say that the substitution is *pure* if the height of the substitution is 1. Dekking [23] introduced the following concept and connected it to the spectrum of the substitution being pure point.

Definition 3.5.5. Let S be an aperiodic, constant length substitution of length q , defined on the alphabet $\mathcal{A} = \{0, 1, \dots, s-1\}$. We say that S admits a *coincidence* if there exists $k \geq 1$ and $j < q^k$ such that

$$S^k(0)_j = S^k(1)_j = \dots = S^k(s-1)_j.$$

In other words, S^k admits a column of identical values and the instruction map at that specific position is constant. We say that the coincidence occurs at order k .

Example 3.5.6. Consider the substitution $S : a \mapsto ab, b \mapsto aa$. This is known as the *period-doubling substitution*. In the first column, both a and b map to a . Therefore the substitution has a coincidence occurring at order 1.

Example 3.5.7. Consider the substitution S defined on $\mathcal{A} = \{a, b, c\}$ as follows:

$$S : a \mapsto bb, b \mapsto cb, c \mapsto ba.$$

The second iterates S^2 gives us

$$a \mapsto cbcb, b \mapsto bacb, c \mapsto cbbb.$$

The last column of the second iterates of all the letters are all b , which means S admits a coincidence and this coincidence occurs at order 2.

Now, we state the theorem of Dekking [42, Thm. 6.6], which connects the pure point spectrum of a given substitution to the coincidence condition.

Theorem 3.7. (*Dekking*) *An aperiodic, pure substitution S of constant length is pure point if and only if it admits a coincidence.*

Another way of characterising the pure point spectrum is that the Fourier coefficients of the correlation measures at extremal points should be periodic with respect to the height function; see [10, Sect.4].

Now let us turn our attention to the Lebesgue component. The central result is the following corollary [10, Cor. 3.16]; it states the following:

Corollary 3.8. *For $w \in \mathcal{K}^*$, the measures $\omega_q * \lambda_w$ are either pure point, pure singular continuous or Lebesgue measure on the torus \mathbb{T} , and they describe the distinct components of the spectrum of S .*

If we are in the Lebesgue measure case, simply write the expression for the k -th Fourier coefficient,

$$\widehat{\lambda}(k) = \int_{\mathbb{T}} e^{-2\pi i k x} d\lambda(x).$$

This expression is zero whenever k is not zero.

3.6 Algorithm

The goal of this section is to summarize the results in the previous sections and provide an algorithm for determining the spectrum of aperiodic substitutions of constant length q ; see [10, Sect. 4] for further information.

Given a substitution of constant length q , one verifies aperiodicity by using Pansiot's lemma 2.12. Aperiodicity is a property that we will assume throughout the algorithm and we let S be an aperiodic substitution of constant length q on an alphabet \mathcal{A} .

We first compute the instruction matrices R_j and the substitution matrix $M_S = \sum_{j \in [0, q)} R_j$. Then from the matrix M_S , we can obtain the Perron–Frobenius vector.

Second, in order to determine the extreme points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution $S \otimes S$ and the coincidence matrix $C_S = \sum_{j \in [0, q)} R_j \otimes R_j$. This gives us a partition of the alphabet \mathcal{A}^2 into its ergodic classes and transient part. One can then apply Proposition 3.5 to determine the hull. One imposes strong semi-positivity of v by computing the eigenvalues of \mathbf{v} , as well as the normalization condition $\sum_{\alpha} v_{\alpha\alpha} u_{\alpha} = 1$, where u_{α} is the Perron–Frobenius vector.

Third, we compute the Fourier coefficients of the correlation measures using the Fourier recursion theorem 3.3, so

$$\widehat{\Sigma}(k) = \frac{1}{q^p} \sum_{j \in [0, q^p)} \left(R_j^{(p)} \otimes R_{j+k}^{(p)} \right) \widehat{\Sigma}(\lfloor j+k \rfloor_p) \quad \text{for } p \in \mathbb{N} \text{ and } k \in \mathbb{Z}.$$

One can solve algebraically for the Fourier coefficient for $k = 1$, like in Example 3.3.2. In general, we have the following equation [10, Eq. 22],

$$\widehat{\Sigma}(1) = \left(q^p \mathbf{I} - \sum_{j \in \Delta_p(1)} R_j^{(p)} \otimes R_{j+1}^{(p)} \right)^{-1} \sum_{j \in [0, q^p) \setminus \Delta_p(1)} R_j^{(p)} \otimes R_{j+1}^{(p)} \widehat{\Sigma}(0). \quad (3.4)$$

Now that the extreme points \mathcal{K}^* and the Fourier coefficients of the correlation measures $\widehat{\Sigma}(k)$ are known, we can then apply Theorem 3.6 to express the spectrum of S in terms of λ_w for $w \in \mathcal{K}^*$. They are categorised as below:

- Lebesgue measure if and only if for all $k \neq 0$, we have $\widehat{\lambda}_w(k) = 0$.

- Pure point measure if and only if $\widehat{\lambda}_w(k)$ is periodic with respect to the height of the substitution.
- Singular continuous if it does not fall into any of the above two categories and $\widehat{\lambda}_w(k) \neq 0$ for some $k \neq 0$.

As mentioned in Corollary 3.8, these measures are all pure.

3.7 Examples

In this section, we use the algorithm provided in the previous section to determine the spectrum of some substitution systems.

We start by completing the analysis of the Thue–Morse substitution of Example 2.2.1.

Example 3.7.1. In Example 3.3.2, we have computed $\widehat{\Sigma}(k)$ for $k = 0, 1, 2$. From Example 3.4.4, we know that \mathcal{K}^* consists of two vectors corresponding to $w = \pm 1$, so $v_1 = e_{aa} + e_{bb} + e_{ab} + e_{ba} = (1, 1, 1, 1)$ and $v_2 = e_{aa} + e_{bb} - e_{ab} - e_{ba} = (1, -1, -1, 1)$, when ordering the vectors in lexicographic order. Now, $\lambda_{v_1} = \sum_{\alpha\beta \in \mathcal{A}^2} \sigma_{\alpha\beta} = \delta_0$, the Dirac measure at 0, see Proposition [42, Prop. 7.5]. For v_2 , we have

$$\widehat{\lambda}_{v_2}(1) = v_2^t \widehat{\Sigma}(1) = -\frac{1}{3} \neq 0 \text{ and } \widehat{\lambda}_{v_2}(2) = v_2^t \widehat{\Sigma}(2) = -\frac{1}{3}.$$

This rules out that the measure is Lebesgue as $\lambda_{v_2}(k) \neq 0$ for some k . Using property 1 of Remark 3.5.2, we know the height of the substitution must be either 1 or 2. However, $\widehat{\Sigma}(5) = \frac{1}{4}(1, 1, 1, 1)$ and we have $\widehat{\lambda}_{v_2}(5) = 0$, therefore, λ_{v_2} cannot be a pure point measure. The Thue–Morse substitution is therefore singular continuous and has a trivial pure point part at the origin.

The following example has been computed in [10, Ex. 4.3], we write out the proof again here as we will compare it with the spectral properties of a Rudin–Shapiro-like sequence in Chapter 4.

Example 3.7.2. Consider the following length two substitution on the alphabet $\mathcal{A} = \{A, B, C, D\}$:

$$S : A \mapsto AB, B \mapsto AC, C \mapsto DB, D \mapsto DC.$$

This is known as the Rudin–Shapiro substitution [10, 6]. The second iterate of the letter A is $ABAC$, which shows that the letter A can be preceded by the letter B and C , hence A has two distinct neighbourhoods. By Pansiot's lemma 2.12, the substitution S is aperiodic.

The instruction matrices R_j for $j \in [0, 2)$ are given by

$$R_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M_S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

As $M^3 > 0$, S is primitive. The Perron–Frobenius vector is $v_{PF} = \frac{1}{4}(1, 1, 1, 1)$.

To determine the extremal points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution $S \otimes S$. The ergodic classes are $E_1 = \{AA, BB, CC, DD\}$, $E_2 = \{AD, BC, CB, DA\}$ and the transient class is given by $T = \{AB, AC, BA, BD, CA, CD, DB, DC\}$. We apply Proposition 3.5 to give

$$\mathbf{v} = \begin{pmatrix} 1 & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & w \\ \frac{1}{2}(1+w) & 1 & w & \frac{1}{2}(1+w) \\ \frac{1}{2}(1+w) & w & 1 & \frac{1}{2}(1+w) \\ w & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & 1 \end{pmatrix} \geq 0. \quad (3.5)$$

We then diagonalise the matrix \mathbf{v} and obtain

$$\mathbf{v}_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-w & 0 & 0 \\ 0 & 0 & 1-w & 0 \\ 0 & 0 & 0 & 2+2w \end{pmatrix},$$

so v is strongly semi-positive if and only if $-1 \leq w \leq 1$. Hence $v \in K(S)$ if and only if \mathbf{v} is of the form of Equation (3.5) with the condition $-1 \leq w \leq 1$. \mathcal{K}^* is given by the vectors

$$v_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

$$v_2 = (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1).$$

As the PF vector is $\frac{1}{4}(1, 1, 1, 1)$ so $\widehat{\Sigma}(0) = \frac{1}{4} \sum_{\gamma \in \mathcal{A}} e_{\gamma\gamma}$. Since this is a length two substitution, $q = 2$ and $\Delta_1(1) = \{1\}$. Equation (3.4) gives us

$$\widehat{\Sigma}(1) = (2\mathbf{I} - R_1 \otimes R_0)^{-1} R_0 \otimes R_1 \widehat{\Sigma}(0) = \frac{1}{8}(0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0),$$

where the bases are ordered lexicographically. Using Theorem 3.3 for $k = 2$ and $p = 1$,

$$\widehat{\Sigma}(2) = \frac{1}{2}(R_0 \otimes R_0 + R_1 \otimes R_1)\widehat{\Sigma}(1) = \frac{1}{8}(1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1).$$

Observe that $\widehat{\Sigma}(1)$ is orthogonal to $\widehat{\Sigma}(2)$. One could then check recursively that $\widehat{\Sigma}(2n+1) = \widehat{\Sigma}(1)$ and $\widehat{\Sigma}(2n) = \widehat{\Sigma}(2)$.

As usual, $\lambda_{v_1} = \delta_0$, and using the Fourier coefficients computed above, $\widehat{\lambda_{v_2}}(k) = 0$ for all $k \neq 0$, so λ_{v_2} is Lebesgue measure. Therefore the Rudin–Shapiro substitution has Lebesgue spectrum.

4

SPECTRUM OF A RUDIN–SHAPIRO-LIKE SEQUENCE

Contents

4.1	Some properties of $(\text{RSL}(n))_{n \geq 0}$	41
4.2	Spectrum of a Rudin–Shapiro-like sequence	43
4.3	Comparison with the Rudin–Shapiro sequence	48

A *digital sequence* is a sequence whose n -th term is defined by some property of the digits of n when written in some bases. They have a wide range of applications, such as in computer science, combinatorics on words and number theory; see [3]. A typical digital sequence is given by the *sum of digits function*, which we denote by $s_k(n)$, which is equal to the sum of digits of the base- k representation of n . When $k = 2$, the sequence $s_2(n)$ counts the number of 1's in the binary representation of n . We refer to the ± 1 's realisation of the sequences we are going to study as the *binary version* of the corresponding sequences, these sequences whose n -term are usually defined by $(-1)^{f(n)}$, where $f(n)$ is a certain function that is defined on the binary representation of n . The binary version of the Thue–Morse sequence of Example 2.2.1 $(t_n)_{n \geq 0}$ (in its realisation as a ± 1 's sequence) is obtained by $t_n = (-1)^{s_2(n)}$, where t_n is the n -th term of the binary Thue–Morse sequence [3, Ex. 5.1.2]. Here we compute the first few terms $(t_n)_{n \geq 0}$:

$$(t_n)_{n \geq 0} = +1, -1, -1, +1, -1, +1, +1, \dots$$

In the previous chapters, we have discussed the Thue–Morse substitution and its spectral type. One can obtain the binary Thue–Morse sequence from its substitution counterpart by mapping a to $+1$ and b to -1 .

Similarly, if we denote by $e_{2;11}(n)$ the number of occurrences of 11's in the binary representation of n , the ± 1 's realisation of the Rudin–Shapiro sequence $(r_n)_{n \geq 0}$ is obtained by $r_n = (-1)^{e_{2;11}(n)}$. One can obtain the binary version of the Rudin–Shapiro sequence of Example 3.7.2 from its substitution by mapping A, B to $+1$ and C, D to -1 .

A *scattered subsequence* of $a_0 a_1 \cdots a_l$ is a word $a_{j_1} a_{j_2} \cdots a_{j_t}$ for some collection of indices $0 \leq j_1 < j_2 < \cdots < j_t \leq l$. Let w be a word over the alphabet $\mathcal{A} = \{0, \dots, k-1\}$. We denote the number of occurrences of w as a scattered subsequence of the base- k representation of n by $\text{sub}_{k;p}(n)$. Thus $\text{sub}_{2;10}(n)$ denotes the number of occurrences of 10 as a scattered subsequence in the binary representation of n . For example, the binary representation of 12 is 1100_2 and the word 1100 has four occurrences of 10 as a scattered subsequence. An *inversion* in a word w is an occurrence of ba as a scattered subsequence of w , where $a, b \in \{0, \dots, k-1\}$ and $b > a$, so the quantity $\text{sub}_{2;10}(n)$ can also be seen as the number of *inversions* in the binary representation of n . For this reason, we will write $\text{inv}_2(n)$ to denote $\text{sub}_{2;10}(n)$.

Recently, Yee, Lafrance and Rampersad [34] introduced the so called *Rudin-Shapiro-like sequence* $(\text{RSL}(n))_{n \geq 0}$, where $\text{RSL}(n) = (-1)^{\text{inv}_2(n)}$. The first few terms of $(\text{RSL}(n))$ are given by

$$(\text{RSL}(n))_{n \geq 0} = +1, +1, -1, +1, +1, -1, +1, +1, \dots$$

The corresponding substitution system S_{RSL} on an alphabet $\mathcal{A} = \{A, B, C, D\}$ is given by

$$S_{\text{RSL}} : A \mapsto AB, B \mapsto CA, C \mapsto BD, D \mapsto DC. \quad (4.1)$$

One can obtain the binary version of the RSL sequence by applying the recoding $A, B \rightarrow +1$ and $C, D \rightarrow -1$ to the infinite sequence $ABCABDABCADCABCA \dots$ which is obtained by iterating S_{RSL} on A .

4.1

Some properties of $(\text{RSL}(n))_{n \geq 0}$

The sequence $(\text{RSL}(n))_{n \geq 0}$ satisfies certain recurrence relations [34, Sect.2]. As before, we denote by t_n the n -th term of the binary Thue–Morse sequence. Then

$$\begin{aligned} \text{RSL}(2n) &= \text{RSL}(n)t_n, \\ \text{RSL}(2n+1) &= \text{RSL}(n). \end{aligned}$$

To see why the above is true, note that if w is the binary representation of n , then $w0$ would be the binary representation of $2n$. The number of occurrences of 10 as a scattered subsequence in $w0$ equals the number of occurrences of 10 as a scattered subsequence in w plus the number of occurrences of 1's in w , so we have

$$\begin{aligned} \text{RSL}(2n) &= (-1)^{\text{inv}_2(2n)} = (-1)^{\text{inv}_2(n) + s_2(n)} \\ &= (-1)^{\text{inv}_2(n)} (-1)^{s_2(n)} = \text{RSL}(n)t_n. \end{aligned}$$

For the second recurrence relation, the binary representation of $2n+1$ is $w1$ and appending 1 in w will not create any new 10 scattered subsequences; hence $\text{RSL}(2n+1) = \text{RSL}(n)$.

As a consequence of the above recurrence relations, one can deduce four more recursion relations [34, Prop. 1]

Proposition 4.1. *The sequence $(\text{RSL}(n))_{n \geq 0}$ satisfies the following recurrence relations:*

$$\begin{aligned}\text{RSL}(4n) &= \text{RSL}(n), \\ \text{RSL}(4n+1) &= \text{RSL}(2n), \\ \text{RSL}(4n+2) &= -\text{RSL}(2n), \\ \text{RSL}(4n+3) &= \text{RSL}(n).\end{aligned}$$

When studying digital sequences, one often looks at the partial sum of the first N terms of the sequence to get an idea of the long-term behaviour. Coquet [18] and Newman [38] investigated the partial sums of the binary Thue–Morse sequence $(t_n)_{n \geq 0}$ taken at multiples of three and found that

$$\sum_{0 \leq n \leq N} t_{3n} = N^{\log_4 3} G_0(\log_4 N) + \frac{1}{3} \eta(N),$$

where G_0 is a bounded, continuous and nowhere differentiable, periodic function with period one, and

$$\eta(N) = \begin{cases} 0 & \text{if } N \text{ is even,} \\ (-1)^{s_2(3N-1)} & \text{if } N \text{ is odd.} \end{cases}$$

Similarly, Brillhart, Erdős and Morton [15] and subsequently Dumont and Thomas [24] studied the partial sums of the first N -th term of the binary Rudin–Shapiro sequence $(r_n)_{n \geq 0}$ and found that

$$\sum_{0 \leq n \leq N} r_n = \sqrt{N} G_1(\log_4 N),$$

where G_1 is a bounded, continuous, nowhere differentiable, periodic function with period one.

In [34], the authors showed that their sequence $(\text{RSL}(n))_{n \geq 0}$ exhibits the same behaviour as the binary Rudin–Shapiro sequence when we look at the partial sums of the first N terms [34, Thm. 2].

Theorem 4.2. *There exists a bounded, continuous, nowhere differentiable, periodic function G with period one such that*

$$\sum_{0 \leq n \leq N} \text{RSL}(n) = \sqrt{N} G(\log_4 N).$$

They also study the properties of the sequence $(\text{RSL}(n))_{n \geq 0}$ from the word combinatorics point of view; see [34, Sect. 5]. At the end of [34], the authors raised the question whether the sequence $(\text{RSL}(n))_{n \geq 0}$ satisfies the following inequality:

$$\sup_{\theta \in \mathbb{R}} \left| \sum_{n < N} \text{RSL}(n) e^{2\pi i n \theta} \right| \leq C N^{\frac{1}{2}}, \quad (4.2)$$

where C is some positive constant. This inequality is referred to as the ‘root- N ’ property. It first appeared as a question of Salem [28, Sect. 2.2] in connection to harmonic analysis. He asked whether one can find a sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ such that the inequality given in Equation (4.2) holds when we replace $\text{RSL}(n)_{n \geq 0}$ by $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$. Golay [31], Rudin [44] and Shapiro [45] gave positive answers to the question by constructing what is now known as the binary Rudin–Shapiro sequence. We shall revisit Rudin’s construction [44] in the next chapter.

In what follows, we will show that the inequality in Equation (4.2) implies that the maximal spectral type has absolutely continuous/Lebesgue component, and the corresponding spectral measure is absolutely continuous. Furthermore, we will employ the algorithm of Bartlett [10] given in the previous chapter to show that the Rudin–Shapiro-like sequence has singular continuous spectrum and hence does not satisfy the inequality in Equation (4.2). This allows us to answer the question raised by the authors at the end of [34].

4.2 Spectrum of a Rudin–Shapiro-like sequence

The connection between the root- N property in Equation (4.2) and the measure defined on the hull of the binary sequences being absolutely continuous was first observed by Allouche and Liardet [2, Sect. 3.3]. Recall, a *correlation function* is given by

$$\eta(m) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n < N} u(n+m) \overline{u(n)},$$

where u is a complex-valued sequence and as a consequence of the definition of the correlation function, $\eta(-m) = \overline{\eta(m)}$. We say a sequence $\{a_n\}_{n \in \mathbb{Z}}$ of complex numbers is *positive definite* on \mathbb{Z} if for any finite sequence $\{z_j\}_{j=1}^k$ of complex numbers, we have $\sum_{j,l=1}^k a_{j-l} z_j \overline{z_l} \geq 0$. By construction, η is a positive

definite sequence on \mathbb{Z} and by the Herglotz–Bochner theorem [6, Thm. 8.6], we know that η is the Fourier transform of a positive measure on \mathbb{T} , which we call the *correlation measure*. By Theorem 2.10, the hull of any primitive substitution under the \mathbb{Z} -action of the shift is strictly ergodic, therefore the sequence u admits a unique correlation measure. We shall use the following description of the unique correlation measure by Queff  lec [42, Prop. 4.9].

Proposition 4.3. *If σ is the unique correlation measure of a fixed point u of a primitive substitution, then σ is the weak- $*$ limit point of the sequence of absolutely continuous measures $R_N \cdot m$, where m is the Lebesgue measure and*

$$R_N(t) = \frac{1}{N} \left| \sum_{n < N} u(n) e^{2\pi i n t} \right|^2.$$

Let us write $\zeta_N = R_N \cdot m$, and suppose the sequence (ζ_N) converges weak- $*$ to a limit ζ . Now, the root- N property in Equation (4.2) gives us that

$$\frac{1}{N} \left| \sum_{n < N} u(n) e^{2\pi i n \theta} \right|^2 \leq C,$$

for some positive constant C . Taking a complex-valued, continuous function g with compact support, we have

$$\zeta(g) \leq \int g \cdot R_N \, dm,$$

and we have $\zeta(g) \leq C \int g \, dm$, which implies absolute continuity.

At the beginning of this chapter, we introduced the Rudin–Shapiro-like substitution and the binary version of it; see Equation (4.1). By the Perron–Frobenius theorem 2.1, the letters in the alphabet appear equally frequently, and as the binary sequences are obtained by applying the reduction map $A, B \rightarrow +1$ and $C, D \rightarrow -1$, both letters ± 1 appear equally frequently as well. We refer to such cases as *balanced weight sequences*.

We are now going to apply Bartlett’s algorithm to prove the following result [16].

Theorem 4.4. *The balanced weight sequence S_{RSL} has singular continuous spectrum.*

Proof. The third iterate of the letter A is $ABCABDAB$, which shows that the letter A can be preceded by C or by D , and that the letter B can be followed

by either C or by D . Hence both A and B have two distinct neighbourhoods and, by Pansiot’s lemma 2.12, the sequence is aperiodic.

The instruction matrices and the substitution matrix can be read off from the substitution rule of Equation (4.1) and are given by

$$R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_{\text{RSL}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

As $M_{\text{RSL}}^3 > 0$, the substitution is primitive. We find the leading eigenvalue of the matrix M_{RSL} is 2 and $v_{\text{PF}} = \frac{1}{4}(1, 1, 1, 1)$ for the corresponding Perron–Frobenius vector.

To determine the extremal points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution $S_{\text{RSL}} \otimes S_{\text{RSL}}$.

$$E_1 : \begin{cases} AA \mapsto AABB \\ BB \mapsto CCAA \\ CC \mapsto BBDD \\ DD \mapsto DDCC \end{cases} \quad E_2 : \begin{cases} AD \mapsto ADBC \\ BC \mapsto CBAD \\ CB \mapsto BCDA \\ DA \mapsto DACB \end{cases} \quad E_3 : \begin{cases} AB \mapsto ACBA \\ AC \mapsto ABBD \\ BA \mapsto CAAB \\ BD \mapsto CDAC \\ CA \mapsto BADB \\ CD \mapsto BDDC \\ DB \mapsto DCCA \\ DC \mapsto DBCD \end{cases}$$

The images of the two letter words under bi-substitution can be formed merely by concatenating the two letter words in the respective class, E_1 , E_2 and E_3 ; this implies we have three ergodic classes and no transient part. The ergodic classes are $E_1 = \{AA, BB, CC, DD\}$, $E_2 = \{AD, BC, CB, DA\}$ and $E_3 = \{AB, AC, BA, BD, CA, CD, DB, DC\}$ and $T = \emptyset$. Since $T = \emptyset$, $P_T = 0$. We apply Proposition 3.5 to give

$$\mathbf{v} = \begin{pmatrix} 1 & w_3 & w_3 & w_2 \\ w_3 & 1 & w_2 & w_3 \\ w_3 & w_2 & 1 & w_3 \\ w_2 & w_3 & w_3 & 1 \end{pmatrix} \geq 0. \quad (4.3)$$

We then diagonalise the matrix \mathbf{v} and obtain:

$$\mathbf{v}_d = \begin{pmatrix} 1 + w_2 + 2w_3 & 0 & 0 & 0 \\ 0 & 1 + w_2 - 2w_3 & 0 & 0 \\ 0 & 0 & 1 - w_2 & 0 \\ 0 & 0 & 0 & 1 - w_2 \end{pmatrix}.$$

Strong semi-positivity is equivalent to w_2 and w_3 satisfying the following three inequalities,

$$1 - w_2 \geq 0, \quad 1 + w_2 + 2w_3 \geq 0, \quad 1 + w_2 - 2w_3 \geq 0,$$

so $v \in \mathcal{K}(S)$ if and only if \mathbf{v} is of the form of Equation (4.3) and w_2, w_3 satisfy the above inequalities. \mathcal{K}^* is given by the vectors

$$\begin{aligned} v_1 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\ v_2 &= (1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1), \\ v_3 &= (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1). \end{aligned}$$

Using Theorem 3.3, we compute some Fourier coefficients of the correlation measure. As the PF vector is $\frac{1}{4}(1, 1, 1, 1)$, so $\widehat{\Sigma}(0) = \frac{1}{4} \sum_{\gamma \in \mathcal{A}} e_{\gamma\gamma}$. As we are dealing with a length two substitution, we have $q = 2$ and $\Delta_1(1) = \{1\}$. By applying Equation (3.4), we have

$$\begin{aligned} \widehat{\Sigma}(1) &= (2\mathbf{I} - R_1 \otimes R_0)^{-1} R_0 \otimes R_1 \widehat{\Sigma}(0) \\ &= \left(0, \frac{1}{6}, 0, \frac{1}{12}, 0, 0, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, 0, 0, \frac{1}{12}, 0, \frac{1}{6}, 0\right). \end{aligned}$$

Using Theorem 3.3, we have

$$\begin{aligned} \widehat{\Sigma}(2) &= \frac{1}{2} \sum_{j \in [0, 2)} (R_j \otimes R_{j+2}) \widehat{\Sigma}([j + 2]_2) \\ &= \frac{1}{2} (R_0 \otimes R_0 + R_1 \otimes R_1) \widehat{\Sigma}(1) \\ &= \left(0, 0, \frac{1}{6}, \frac{1}{12}, \frac{1}{6}, 0, \frac{1}{12}, 0, 0, \frac{1}{12}, 0, \frac{1}{6}, \frac{1}{12}, \frac{1}{6}, 0, 0\right), \\ \widehat{\Sigma}(3) &= \frac{1}{2} \sum_{j \in [0, 2)} (R_j \otimes R_{j+3}) \widehat{\Sigma}([j + 3]_1) \\ &= \frac{1}{2} (R_0 \otimes R_1 \widehat{\Sigma}(1) + R_1 \otimes R_0 \widehat{\Sigma}(2)) \\ &= \left(\frac{1}{6}, \frac{1}{24}, \frac{1}{24}, 0, \frac{1}{24}, \frac{1}{6}, 0, \frac{1}{24}, \frac{1}{24}, 0, \frac{1}{6}, \frac{1}{24}, 0, \frac{1}{24}, \frac{1}{24}, \frac{1}{6}\right), \end{aligned}$$

$$\begin{aligned}
\widehat{\Sigma}(4) &= \frac{1}{2} \sum_{j \in [0,2)} (R_j \otimes R_{j+4}) \widehat{\Sigma}(\lfloor j+4 \rfloor_1) \\
&= \frac{1}{2} (R_0 \otimes R_0 + R_1 \otimes R_1) \widehat{\Sigma}(2) \\
&= \left(0, \frac{1}{6}, 0, \frac{1}{12}, 0, 0, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, 0, 0, \frac{1}{12}, 0, \frac{1}{6}, 0 \right),
\end{aligned}$$

$$\begin{aligned}
\widehat{\Sigma}(5) &= \frac{1}{2} \sum_{j \in [0,2)} (R_j \otimes R_{j+5}) \widehat{\Sigma}(\lfloor j+5 \rfloor_1) \\
&= \frac{1}{2} (R_0 \otimes R_1 \widehat{\Sigma}(2) + R_1 \otimes R_0 \widehat{\Sigma}(3)) \\
&= \left(\frac{1}{48}, 0, \frac{1}{8}, \frac{5}{48}, \frac{1}{8}, \frac{1}{48}, \frac{5}{48}, 0, 0, \frac{5}{48}, \frac{1}{48}, \frac{1}{8}, \frac{5}{48}, \frac{1}{8}, 0, \frac{1}{48} \right).
\end{aligned}$$

As usual, $\lambda_{v_1} = \sum_{\alpha\beta \in \mathcal{A}^2} \sigma_{\alpha\beta} = \delta_0$, which gives rise to the pure point component, via Theorem 3.6. Using the computed values of $\widehat{\Sigma}(k)$, we have for λ_{v_2}

$$\begin{aligned}
\widehat{\lambda}_{v_2}(1) &= v_2^t \widehat{\Sigma}(1) = -\frac{1}{3}, \quad \widehat{\lambda}_{v_2}(2) = -\frac{1}{3}, \quad \widehat{\lambda}_{v_2}(3) = \frac{1}{3}, \quad \widehat{\lambda}_{v_2}(4) = -\frac{1}{3}, \\
\widehat{\lambda}_{v_2}(5) &= 0.
\end{aligned}$$

For λ_{v_3} , we have

$$\begin{aligned}
\widehat{\lambda}_{v_3}(1) &= v_3^t \widehat{\Sigma}(1) = -\frac{1}{3}, \quad \widehat{\lambda}_{v_3}(2) = -\frac{1}{3}, \quad \widehat{\lambda}_{v_3}(3) = \frac{2}{3}, \quad \widehat{\lambda}_{v_3}(4) = -\frac{1}{3}, \\
\widehat{\lambda}_{v_3}(5) &= -\frac{1}{3}.
\end{aligned}$$

We can observe from the computed values above that $\widehat{\lambda}_{v_2}(k)$ and $\widehat{\lambda}_{v_3}(k)$ do not vanish at all positions $k \neq 0$, which proves that there are no Lebesgue components.

Since the substitution S_{RSL} is a substitution of length two on an alphabet of four letters, we have $q = 2$ and the height h is $1 \leq h \leq 4$. Using property 3 of Remark 3.5.2, the numbers that are coprime to q in this case are 1 and 3. Take $h = 3$; we want to see if this height partitions the alphabet. We look at $S_{\text{RSL}}^4(A) = ABCABDABCADCABCA$, and look at the letters at positions 0, 1 and 2 mod 3 and observe the following

$$\begin{aligned}
0 \bmod 3 &: A, A, A, A, A, \\
1 \bmod 3 &: B, B, B, D, B, \\
2 \bmod 3 &: C, D, C, C, C.
\end{aligned}$$

Since the letter D can be identified in both 1 and 2 mod 3, this does not form a partition of the alphabet, so the height in this case is $h = 1$. In order to give rise to a non-trivial pure point component, the coefficients of $\widehat{\lambda}_{v_2}(k)$ and $\widehat{\lambda}_{v_3}(k)$ would have to be 1-periodic, which they are clearly not from the computed values. Hence the pure point component is entirely supported by the Dirac measure δ_0 . The two measures $\widehat{\lambda}_{v_2}(k)$ and $\widehat{\lambda}_{v_3}(k)$ are thus neither absolutely continuous nor show the necessary periodicity to contribute to the pure point part. We thus conclude these two measures have to be singular continuous. We therefore have a purely singular continuous spectrum in the balanced weight case. \square

As we discussed earlier, the inequality in Equation (4.2) implies absolute continuity of the spectrum. Showing that the balanced weight Rudin–Shapiro-like sequence is purely singular continuous and does not have any absolutely continuous component in the spectrum implies that it does not satisfy the root- N property. Thus, we have answered the question raised at the end of [34].

4.3

Comparison with the Rudin–Shapiro sequence

We shall finish off this chapter with a brief comparison of the spectral properties with the Rudin–Shapiro sequence. The following result about the Rudin–Shapiro sequence is well-known; see [6, Ch. 10.2] and references therein for background and details.

Proposition 4.5. *The balanced weight Rudin–Shapiro sequence has Lebesgue spectrum.*

We refer to Example 3.7.2 and [10, Ex. 4.3] to see how Bartlett’s algorithm can be employed to show the above result.

Both the Rudin–Shapiro and the Rudin–Shapiro-like sequence are based on (four-letter) substitutions of constant length $q = 2$ (and a subsequent reduction to a balanced two-letter sequence), and superficially look quite similar, including sharing the same behaviour of partial sums. The ergodic classes E_1 and E_2 of both substitutions contain exactly the same elements: $E_1 = \{AA, BB, CC, DD\}$, $E_2 = \{AD, BC, CB, DA\}$. The elements that form

the transient class of the Rudin–Shapiro sequence are exactly the same elements that form the third ergodic class of the Rudin–Shapiro-like sequence, namely $\{AB, AC, BA, BD, CA, CD, DB, DC\}$. However, the values obtained from the Fourier transform of the correlation measures differ between these two systems. Hence we have two structurally different systems that exhibit similar arithmetic behaviour.

5

GENERALISATION OF RUDIN'S METHOD: REAL WEIGHTS

Contents

5.1	Introduction	52
5.2	The Rudin–Shapiro sequence revisited	54
5.3	Modifying Rudin’s construction	56

5.1 Introduction

In this chapter, we are interested in substitution systems that feature absolutely continuous or Lebesgue spectra. In Example 3.7.2, we saw that the balanced weights (± 1) Rudin–Shapiro sequence has absolutely continuous spectrum. This result is well-known and we refer the readers to [6, Sect. 10.2] and [10, Ex. 4.3] for details of the proof. As mentioned in the previous chapter, the binary version of the Rudin–Shapiro sequence was introduced in [31, 45, 44] in answer to a question raised by Salem [28, Sect. 2.3] in the context of harmonic analysis; see also [6, Sect. 4.7.1].

This sequence in its balanced weight case is a paradigm of a substitution-based structure with absolutely continuous spectrum. Some generalisations of the Rudin–Shapiro sequence were provided in [3], but to date relatively few examples of substitution-based sequences of this type are known explicitly.

In [29], a systematic generalisation of the Rudin–Shapiro system to higher-dimensional substitutions was derived. It employs Hadamard matrices (matrices with elements ± 1 whose rows are mutually orthogonal). The underlying substitutions are constant length substitutions on a finite alphabet \mathcal{A} , based on arrangements of letters on the (hyper)cubic lattice \mathbb{Z}^d . Letters in the alphabet are paired, so that for each letter $a \in \mathcal{A}$, there is a twin letter $\bar{a} \in \mathcal{A}$, with $\bar{\bar{a}} = a$ and $\bar{a} \neq a$. Let us illustrate the above concepts using one of Frank's examples [29, Fig. 1].

Example 5.1.1. Take $d = 2$ and we consider a \mathbb{Z}^2 -substitution S on the alphabet $\mathcal{A} = \{A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D}\}$. The substitution S is given as follows:

$$\begin{aligned} S(A) &= \begin{array}{c|c} C & \bar{D} \\ \hline A & B \end{array} & S(B) &= \begin{array}{c|c} \bar{C} & D \\ \hline A & B \end{array} & S(C) &= \begin{array}{c|c} C & D \\ \hline A & \bar{B} \end{array} & S(D) &= \begin{array}{c|c} C & D \\ \hline \bar{A} & B \end{array} \\ S(\bar{A}) &= \begin{array}{c|c} \bar{C} & D \\ \hline \bar{A} & \bar{B} \end{array} & S(\bar{B}) &= \begin{array}{c|c} C & \bar{D} \\ \hline \bar{A} & \bar{B} \end{array} & S(\bar{C}) &= \begin{array}{c|c} \bar{C} & \bar{D} \\ \hline \bar{A} & B \end{array} & S(\bar{D}) &= \begin{array}{c|c} \bar{C} & \bar{D} \\ \hline A & \bar{B} \end{array} \end{aligned}$$

If we give the barred letters an underlying value of -1 and the unbarred version an underlying value of $+1$, a *symbol matrix* is similar to the substitution matrix except it encodes the barred and unbarred letters in the images of the

substitution, and one only specifies either the barred or the unbarred versions of the substitution. The author proved the following result [29, Thm. 4.1], in which \mathbb{X} denotes the hull of the substitution, μ the corresponding invariant measure, H_D the pure point spectrum, and $Z(f)$ the cyclic subspace associated to a function $f \in L^2(\mathbb{X}, \mu)$.

Theorem 5.1. *Let $(\mathbb{X}, \mathbb{Z}^d, \mu)$ be a dynamical system associated to an aperiodic \mathbb{Z}^d -substitution subject to the conditions thatⁱ*

- *each letter in the alphabet \mathcal{A} is only allowed to appear in the position given by its underlying number (so the images of letters under the substitution differ only in the number and/or position of the bars that distinguish paired letters);*
- *paired letters are substituted by corresponding paired blocks;*
- *the symbol matrix of the corresponding substitution is a Hadamard matrix.*

Then there exist functions $f_1, \dots, f_K \in L^2(\mathbb{X}, \mu)$, each with spectral measure equal to Lebesgue measure, such that

$$L^2(\mathbb{X}, \mu) = H_D \oplus Z(f_1) \oplus \dots \oplus Z(f_K).$$

The substitution S of Example 5.1.1 satisfies the first two conditions of Theorem 5.1, as each letter in the alphabet only appears at the position designated in the image under the substitution, and once we know the unbarred version, the barred version follows. In the symbol matrix of such a substitution, we usually specify either the barred version or the unbarred version only. So for the substitution S of Example 5.1.1, we consider the unbarred version, and the symbol matrix is

$$\begin{pmatrix} +1 & +1 & +1 & -1 \\ +1 & +1 & -1 & +1 \\ +1 & -1 & +1 & +1 \\ -1 & +1 & +1 & +1 \end{pmatrix}.$$

The symbol matrix is clearly Hadamard, hence the substitution of Example 5.1.1 satisfies all three conditions of Theorem 5.1, hence it must be a \mathbb{Z}^2 substitution with Lebesgue components.

ⁱWe refer to the original article for more details on these conditions.

In this chapter, we provide further examples of substitution-based structures with Lebesgue spectrum. These systems do not satisfy the last condition of Theorem 5.1, although they still appear to have a close relationship to Hadamard matrices and their complex analogues. Our approach is based on modifying and extending the original construction of Rudin [44]. As such, we look at sequences $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ that satisfy the property

$$\sup_{|x|=1} \left| \sum_{n=1}^N \varepsilon_n x^n \right| \leq C N^{\frac{1}{2}}, \quad (5.1)$$

for some positive constant C , where the supremum is taken over complex numbers of unit modulus. This bound on the growth of the exponential sums implies that the corresponding spectrum is absolutely continuous; see Section 4.2 or compare [2, 16].

We start by revisiting the recurrence relations that give rise to the Rudin–Shapiro sequence [44]. We then generalise this approach in Section 5.3 by introducing a sequence of signs in the recurrence relations. This results in substitution-based structures in which the underlying substitutions are of length 2^k for $k \in \mathbb{N}$.

5.2 The Rudin–Shapiro sequence revisited

Let us briefly review Rudin's original construction of the Rudin–Shapiro (RS) sequence. For details of the construction, see [44].

We start by defining two sequences of polynomials, $(P_k(x))_{k \in \mathbb{N}_0}$ and $(Q_k(x))_{k \in \mathbb{N}_0}$, where P_k and Q_k both have degree 2^k . They are determined by the initial choices $P_0(x) = Q_0(x) = x$ together with the recurrence relations

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - x^{2^k} Q_k(x). \end{aligned} \quad (5.2)$$

It is clear from Equation (5.2) that the first 2^k terms of $P_{k+1}(x)$ and of $Q_{k+1}(x)$ coincide with those of $P_k(x)$, and that their remaining terms differ by a sign. By construction, $P_k(x)$ is of the form

$$P_k(x) = \sum_{n=1}^{2^k} \varepsilon_n x^n, \quad (5.3)$$

where each coefficient ε_i is either -1 or 1 , so we can define a binary sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ from the coefficients. This is the binary RS sequence. For example, for $k = 3$, we have the polynomial

$$P_3(x) = x + x^2 + x^3 - x^4 + x^5 + x^6 - x^7 + x^8,$$

from which we read off the sequence $111\bar{1}\bar{1}1\bar{1}1$, where here (and henceforth) we use the convention that $\bar{1} = -1$.

If $a_k = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_{2^k} \in \{\pm 1\}^{2^k}$ denotes the word of length 2^k of coefficients of $P_k(x)$, and b_k denotes the corresponding word for $Q_k(x)$, then the recurrence relations of Equation (5.2) correspond to the concatenation relations

$$\begin{aligned} a_{k+1} &= a_k b_k, \\ b_{k+1} &= a_k \bar{b}_k, \end{aligned} \tag{5.4}$$

on words in the two-letter alphabet $\{1, \bar{1}\}$, with initial values $a_0 = b_0 = 1$.

The concatenation relations from Equation (5.4) can be seen to correspond to the substitution rule $A \mapsto AB$, $B \mapsto A\bar{B}$ on the four-letter alphabet $\{A, B, \bar{A}, \bar{B}\}$, which upon completion to a four-letter substitution rule becomes

$$S_+ : \quad A \mapsto AB, \quad B \mapsto A\bar{B}, \quad \bar{A} \mapsto \bar{A}\bar{B}, \quad \bar{B} \mapsto \bar{A}B, \tag{5.5}$$

so that the ‘bar’ operation is compatible with the substitution; see [5] for more on substitutions that feature a ‘bar-swap symmetry’ of this kind. This substitution is often referred to as the four-letter RS substitution rule. Clearly, by induction, this rule gives rise to the concatenation relations

$$\begin{aligned} A_{k+1} &= A_k B_k, \\ B_{k+1} &= A_k \bar{B}_k, \end{aligned}$$

which have the same structure as Equation (5.4), but work on the four-letter alphabet $\{A, B, \bar{A}, \bar{B}\}$ instead of the two-letter alphabet $\{1, \bar{1}\}$. The connection between the two is provided by the map

$$\varphi : \begin{cases} A, B \mapsto 1, \\ \bar{A}, \bar{B} \mapsto \bar{1}. \end{cases} \tag{5.6}$$

Iterating the substitution rule S_+ (defined by Equation (5.5)) on the initial letter A gives

$$\begin{aligned} A &\mapsto AB \mapsto AB A\bar{B} \mapsto AB A\bar{B} AB \bar{A}B \\ &\mapsto AB A\bar{B} AB \bar{A}B AB A\bar{B} \bar{A}\bar{B} \bar{A}\bar{B} \mapsto \cdots \longrightarrow w_+, \end{aligned}$$

which converges (in the local topology) to an infinite fixed point word w_+ .ⁱⁱ We denote the corresponding hull, which is the closure of the orbit of w_+ under the shift map, by \mathbb{X}_+ . The binary RS sequence is then recovered as the image of w_+ under the factor map φ of Equation (5.6), which reproduces the sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$. Note that there is no two-letter substitution rule for this sequence, unless you work with a staggered substitution with different rules for even and odd positions along the word; see [6, Sect. 4.7.1].

The main ingredient in Rudin's proof [44] of the root- N property of Equation (5.1) for the binary sequence is the parallelogram law,

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2, \quad (5.7)$$

where $\alpha, \beta \in \mathbb{C}$, and this will also be the case in our generalisations discussed below. Note that this implies that the consequences for spectral properties specifically apply to the binary sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$; the argument does not directly provide information about the spectral properties of the underlying four-letter sequence obtained from the substitution rule of Equation (5.5).

5.3 Modifying Rudin's construction

Let us now introduce some modifications to the original construction of Rudin, and show that our newly-derived recurrence relations still satisfy the root- N property of Equation (5.1). Following this, we compute some concrete examples and derive the corresponding substitution systems, in the same way as for the RS sequence above.

We again work with two sequences of polynomials $(P_k(x))_{k \in \mathbb{N}_0}$ and $(Q_k(x))_{k \in \mathbb{N}_0}$, with $P_0(x) = Q_0(x) = x$. By introducing additional signs $\sigma_k \in \{\pm 1\}$ in the recurrence relations of Equation (5.2), we consider

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + \sigma_k x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - \sigma_k x^{2^k} Q_k(x), \end{aligned} \quad (5.8)$$

for $k \in \mathbb{N}_0$. At this stage, we do not yet specify the values of σ_k .

ⁱⁱHere and below we use the initial letter A to construct a fixed point sequence w . There will always be a second fixed point sequence, which due to the bar-swap symmetry of our substitutions is just \overline{w} , which can be obtained by iterating S_+ on the initial letter \overline{A} .

Clearly, the RS case corresponds to the choice $\sigma_k = 1$ for all $k \in \mathbb{N}_0$. If instead one chooses $\sigma_k = -1$ for all $k \in \mathbb{N}_0$, the recurrence relations in Equation (5.8) becomes

$$\begin{aligned} P_{k+1}(x) &= P_k(x) - x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) + x^{2^k} Q_k(x). \end{aligned} \quad (5.9)$$

If $a_k = \varepsilon_1 \cdots \varepsilon_{2^k} \in \{\pm 1\}^{2^k}$ denotes the word of length 2^k of coefficients of $P_k(x)$ and b_k denotes the corresponding word for $Q_k(x)$, then the recurrence relations of Equation (5.9) correspond to the concatenation relations

$$\begin{aligned} a_{k+1} &= a_k \overline{b_k}, \\ b_{k+1} &= a_k b_k, \end{aligned} \quad (5.10)$$

on words in the two-letter alphabet $\{1, -1\}$, with initial values $a_0 = b_0 = 1$.

The concatenation relations from Equation (5.10) can be seen to correspond to the substitution below

$$S_-: \quad A \mapsto A\overline{B}, \quad B \mapsto AB, \quad \overline{A} \mapsto \overline{A}B, \quad \overline{B} \mapsto \overline{A}\overline{B}. \quad (5.11)$$

Its one-sided fixed point w_- , obtained by iterating S_- on the letter A ,

$$\begin{aligned} A \mapsto A\overline{B} \mapsto A\overline{B}\overline{A}\overline{B} \mapsto A\overline{B}\overline{A}\overline{B}\overline{A}B\overline{A}\overline{B} \\ \mapsto A\overline{B}\overline{A}\overline{B}\overline{A}B\overline{A}\overline{B}\overline{A}B\overline{A}B\overline{A}\overline{B}\overline{A}\overline{B} \mapsto \cdots \longrightarrow w_-, \end{aligned}$$

gives rise to the hull \mathbb{X}_- . Using Bartlett's algorithm in Section 3.6, we now prove the following result regarding the spectral type of S_- .

Proposition 5.2. *The balanced weight sequence S_- has Lebesgue spectrum.*

Proof. The third iterate of the letter A is $A\overline{B}\overline{A}\overline{B}\overline{A}B\overline{A}\overline{B}$, which shows that the letter \overline{A} can be preceded by \overline{B} or B . Hence the letter \overline{A} has two distinct neighbourhoods, so, by Pansiot's lemma 2.12, the sequence is aperiodic.

The instruction matrices and the substitution matrix can be read off from the substitution rule of Equation (5.11) and are given by

$$R_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{S_-} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As $M_{S_-}^3 > 0$, the substitution is primitive. We find the leading eigenvalue of the matrix M_{S_-} is 2 and $v_{\text{PF}} = \frac{1}{4}(1, 1, 1, 1)$ for the corresponding Perron–Frobenius vector.

To determine the extremal points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution $S_- \otimes S_-$.

$$E_1 : \begin{cases} A A \mapsto A A \bar{B} \bar{B} \\ B B \mapsto A A B B \\ \bar{B} \bar{B} \mapsto \bar{A} \bar{A} \bar{B} \bar{B} \\ \bar{A} \bar{A} \mapsto \bar{A} \bar{A} B B \end{cases} \quad E_2 : \begin{cases} A \bar{A} \mapsto A \bar{A} \bar{B} B \\ B \bar{B} \mapsto A \bar{A} B \bar{B} \\ \bar{B} B \mapsto \bar{A} \bar{A} \bar{B} B \\ \bar{A} A \mapsto \bar{A} \bar{A} \bar{B} B \end{cases} \quad T : \begin{cases} A B \mapsto A A \bar{B} B \\ A \bar{B} \mapsto A \bar{A} \bar{B} \bar{B} \\ B A \mapsto A A B \bar{B} \\ B \bar{A} \mapsto A \bar{A} B B \\ \bar{B} A \mapsto \bar{A} \bar{A} \bar{B} \bar{B} \\ \bar{B} \bar{A} \mapsto \bar{A} \bar{A} \bar{B} B \\ \bar{A} B \mapsto \bar{A} \bar{A} B B \\ \bar{A} \bar{B} \mapsto \bar{A} \bar{A} B \bar{B} \end{cases}$$

We have two ergodic classes and one transient class and they are given by $E_1 = \{A A, B B, \bar{B} \bar{B}, \bar{A} \bar{A}\}$, $E_2 = \{A \bar{A}, B \bar{B}, \bar{B} B, \bar{A} A\}$ and $T = \{A B, A \bar{B}, B A, B \bar{A}, \bar{B} A, \bar{B} \bar{A}, \bar{A} B, \bar{A} \bar{B}\}$. We apply Proposition 3.5, with

$$\mathbf{v} = \begin{pmatrix} 1 & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & w \\ \frac{1}{2}(1+w) & 1 & w & \frac{1}{2}(1+w) \\ \frac{1}{2}(1+w) & w & 1 & \frac{1}{2}(1+w) \\ w & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & 1 \end{pmatrix} \geq 0. \quad (5.12)$$

We then diagonalise the matrix \mathbf{v} and obtain

$$\mathbf{v}_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-w & 0 & 0 \\ 0 & 0 & 1-w & 0 \\ 0 & 0 & 0 & 2+2w \end{pmatrix},$$

so v is strongly semi-positive if and only if $-1 \leq w \leq 1$. Hence $v \in \mathcal{K}(S)$ if and only if \mathbf{v} is of the form of Equation (5.12) with the condition $-1 \leq w \leq 1$. \mathcal{K}^* is given by the vectors

$$\begin{aligned} v_1 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \\ v_2 &= (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1). \end{aligned}$$

As the PF vector is $\frac{1}{4}(1, 1, 1, 1)$ so $\widehat{\Sigma}(0) = \frac{1}{4} \sum_{\gamma \in \mathcal{A}} e_{\gamma\gamma}$. Since this is a length two substitution, $q = 2$ and $\Delta_1(1) = \{1\}$. Equation (3.4) gives us

$$\widehat{\Sigma}(1) = (2\mathbf{I} - R_1 \otimes R_0)^{-1} R_0 \otimes R_1 \widehat{\Sigma}(0) = \frac{1}{8}(0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0).$$

Using Theorem 3.3 for $k = 2$ and $p = 1$,

$$\widehat{\Sigma}(2) = \frac{1}{2}(R_0 \otimes R_0 + R_1 \otimes R_1) \widehat{\Sigma}(1) = \frac{1}{8}(1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1).$$

Observe that $\widehat{\Sigma}(1)$ is orthogonal to $\widehat{\Sigma}(2)$. By direct computation and application of the Fourier recursion theorem 3.3, one can establish the following relations.

$$\begin{cases} \widehat{\Sigma}(2n) = \frac{1}{2}(R_0 \otimes R_0 + R_1 \otimes R_1) \widehat{\Sigma}(n) = \frac{1}{2}C_{S_-} \widehat{\Sigma}(n), \\ \frac{1}{2}C_{S_-} \widehat{\Sigma}(1) = \frac{1}{2}C_{S_-} \widehat{\Sigma}(2) = \widehat{\Sigma}(2), \end{cases}$$

$$\begin{cases} \widehat{\Sigma}(2n+1) = \frac{1}{2} \left(R_0 \otimes R_1 \widehat{\Sigma}(n) + R_1 \otimes R_0 \widehat{\Sigma}(2n+1) \right), \\ \frac{1}{2} \left(R_0 \otimes R_1 \widehat{\Sigma}(1) + R_1 \otimes R_0 \widehat{\Sigma}(2) \right) = \widehat{\Sigma}(1), \\ \frac{1}{2} \left(R_0 \otimes R_1 \widehat{\Sigma}(2) + R_1 \otimes R_0 \widehat{\Sigma}(1) \right) = \widehat{\Sigma}(1). \end{cases}$$

Then, we can proceed by induction and obtain the identities $\widehat{\Sigma}(2n+1) = \widehat{\Sigma}(1)$ and $\widehat{\Sigma}(2n) = \widehat{\Sigma}(2)$ for $n \neq 0$.

As usual, $\lambda_{v_1} = \delta_0$, and using the Fourier coefficients computed above, $\widehat{\lambda}_{v_2}(2n) = v_2^t \widehat{\Sigma}(2n) = v_2^t \widehat{\Sigma}(2) = 0$ and $\widehat{\lambda}_{v_2}(2n+1) = v_2^t \widehat{\Sigma}(2n+1) = v_2^t \widehat{\Sigma}(1) = 0$, so λ_{v_2} is Lebesgue measure. Therefore the balanced weight sequence S_- has Lebesgue spectrum. \square

It is easy to verify that $\mathbb{X}_+ \neq \mathbb{X}_-$, which is a consequence of Proposition 2.4. The hull of a substitution is equal to the closure of the local indistinguishability (LI) class, and two words u and v are LI if each finite subword of u is also a subword of v and vice versa. Since there are subwords of length six in w_+ (such as $B A B A \overline{B} \overline{A}$) which do not occur as subwords of w_- , and vice versa, the hulls \mathbb{X}_+ and \mathbb{X}_- cannot be equal. Indeed, the same holds true for the corresponding binary sequences and their hulls $\mathbb{Y}_+ := \varphi(\mathbb{X}_+)$ and $\mathbb{Y}_- := \varphi(\mathbb{X}_-)$ with φ defined in Equation (5.6). For example, $\varphi(B A B A \overline{B} \overline{A}) = 1111\overline{1}\overline{1}$ is a subword of $\varphi(w_+)$ but not of $\varphi(w_-)$, as $ABAB$ does not appear in either w_+ or w_- . Although \mathbb{X}_+ and \mathbb{X}_- are different, and so are their corresponding binary hulls \mathbb{Y}_+ and \mathbb{Y}_- , the quaternary hull \mathbb{X}_+ and the binary hull \mathbb{Y}_+ (and

also \mathbb{X}_- and \mathbb{Y}_-) are intimately connected by the concept of (mutual) local derivability, which was introduced in [8].

Definition 5.3.1. A *pattern* P in Euclidean space \mathbb{R}^d is a non-empty set of non-empty subsets of \mathbb{R}^d . We refer to the elements of P as the *fragments* of P .

If P is a pattern in \mathbb{R}^d , and $A \subset \mathbb{R}^d$, we use the notation $P \sqcap A$ to denote the subset of P that consists of all fragments of P that intersect A , so $P \sqcap A = \{p \in P : p \cap A \neq \emptyset\}$.

Definition 5.3.2. A pattern P' in \mathbb{R}^d is said to be *locally derivable* from a pattern P in \mathbb{R}^d , when a compact neighbourhood $K \subset \mathbb{R}^d$ of 0 exists such that whenever $(P - x) \sqcap K = (P - y) \sqcap K$ holds for $x, y \in \mathbb{R}^d$, one also has $(P' - x) \sqcap \{0\} = (P' - y) \sqcap \{0\}$. Two patterns P and P' are called *mutually locally derivable* (MLD) from each other when P is locally derivable from P' and P' is locally derivable from P .

Local derivability gives a (formal) rule to construct the part of P' around a given point solely from the knowledge of the K -neighbourhood of that point in P . When two patterns P and P' are MLD, there exists a bijective, local rule that maps part of P to P' and vice versa.

Adapting the notion of MLD to substitution systems, observe that φ induces a bijection between the hulls, so \mathbb{X}_+ and \mathbb{Y}_+ (and also \mathbb{X}_- and \mathbb{Y}_-) are mutually locally derivable, and the corresponding four-letter and two-letter dynamical systems (under the shift action) are topologically conjugate, as φ commutes with the shift action; compare [6, Rem. 4.11]. This can, for instance, be seen by realising that the subword 1111, which occurs in both $\varphi(w_+)$ and $\varphi(w_-)$ with bounded gaps, has the unique preimage $BABA$ in both w_+ and w_- .

We now show that the generalisation by the factors σ_k in Equation (5.8) still produces sequences that satisfy the root- N property.

Proposition 5.3. *The sequence of coefficients $(\varepsilon_n)_{n \in \mathbb{N}}$ of the functions P_k , $k \in \mathbb{N}_0$, defined by the recurrence relations of Equation (5.8), satisfies the root- N property of Equation (5.1).*

Proof. Suppose $|x| = 1$. By the recurrence relations (5.8), we have

$$|P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 = |P_k(x) + \sigma_k x^{2^k} Q_k(x)|^2 + |P_k(x) - \sigma_k x^{2^k} Q_k(x)|^2.$$

Applying the parallelogram law (5.7), we find that

$$|P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 = 2(|P_k(x)|^2 + |Q_k(x)|^2).$$

Since $|P_0(x)|^2 + |Q_0(x)|^2 = 2$, we can conclude that

$$|P_k(x)|^2 + |Q_k(x)|^2 = 2^{k+1},$$

and hence

$$|P_k(x)| \leq 2^{\frac{k+1}{2}}. \quad (5.13)$$

This proves the root- N property for $N = 2^k$.

In order to tackle the case when N is not necessarily a power of 2, we define partial sums of P_k and Q_k as follows,

$$P_{k|m}(x) = \sum_{n=1}^m \varepsilon_n x^n, \quad Q_{k|m}(x) = \sum_{n=1}^m \gamma_n x^n,$$

where $2^{k-1} < m \leq 2^k$, $k \in \mathbb{N}_0$, and where $\varepsilon_n, \gamma_n \in \{\pm 1\}$ are the corresponding coefficients. We now show that these satisfy

$$|P_{k|m}(x)| \leq G 2^{\frac{k}{2}} \quad \text{and} \quad |Q_{k|m}(x)| \leq G 2^{\frac{k}{2}} \quad (5.14)$$

for all $|x| = 1$ and $k \in \mathbb{N}_0$, where $G = 2 + 2^{1/2}$.

The proof proceeds by induction. The above estimates are obviously true for $k = 0$. Suppose that they hold for some $k \in \mathbb{N}_0$, and consider an integer m with $2^k < m \leq 2^{k+1}$. By using the triangle inequality together with Equations (5.13) and (5.14), we obtain

$$|P_{k+1|m}(x)| \leq |P_k(x)| + |Q_{k|m-2^k}(x)| \leq 2^{\frac{k+1}{2}} + G 2^{\frac{k}{2}} = G 2^{\frac{k+1}{2}},$$

which establishes Equation (5.14) for $k + 1$. The same argument clearly works for $Q_{k+1|m}(x)$.

To complete the proof, suppose that $2^{k-1} < N \leq 2^k$. By Equation (5.14), we have

$$|P_{k|N}(x)| \leq (2 + 2^{\frac{1}{2}}) 2^{\frac{k}{2}} \leq 2(1 + 2^{\frac{1}{2}}) N^{\frac{1}{2}},$$

which shows that the root- N property holds. \square

Corollary 5.4. *Whatever the choice of the signs $\sigma_k \in \{\pm 1\}$ in Equation (5.8), the corresponding sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ is balanced.*

Proof. The average value of the first N coefficients is given by

$$\frac{1}{N} \sum_{n=1}^N \varepsilon_n = \frac{1}{N} P_{k|N}(1),$$

where $2^{k-1} < N \leq 2^k$. By Proposition 5.3, this satisfies

$$\left| \frac{1}{N} \sum_{n=1}^N \varepsilon_n \right| \leq 2(1 + 2^{\frac{1}{2}})N^{-\frac{1}{2}},$$

and therefore the average value tends to 0 as $N \rightarrow \infty$. \square

As mentioned earlier, the root- N property implies that the spectral measure is absolutely continuous for the binary sequence. We now consider some examples.

Example 5.3.3. Let us start with the choice $\sigma_k = (-1)^{k+1}$, so the signs in the recurrence relations for the polynomials alternate, and we have

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + (-1)^{k+1} x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - (-1)^{k+1} x^{2^k} Q_k(x), \end{aligned}$$

for $k \in \mathbb{N}_0$. We could now read off the corresponding substitution rule just as we did for the RS substitution, but this case is more complicated because of the alternating signs. If we try to read off the substitution rule by studying $P_{k+1}(x), Q_{k+1}(x)$ at each step, we would quickly realise that the letter A has two distinct images $A\bar{B}$ and $\bar{A}B$ which makes it not a well-defined substitution. One way to overcome this problem is to look at two consecutive steps at once,

$$\begin{aligned} P_{k+2}(x) &= P_k(x) + (-1)^{k+1} x^{2^k} Q_k(x) + (-1)^{k+2} x^{2 \cdot 2^k} P_k(x) + x^{3 \cdot 2^k} Q_k(x), \\ Q_{k+2}(x) &= P_k(x) + (-1)^{k+1} x^{2^k} Q_k(x) - (-1)^{k+2} x^{2 \cdot 2^k} P_k(x) - x^{3 \cdot 2^k} Q_k(x). \end{aligned} \tag{5.15}$$

Choosing k to be even (which corresponds to the case we are interested in, since our recursion starts with $k = 0$) and associating letters A and B , and their counterparts \bar{A} and \bar{B} , to the sequences corresponding to P and Q , we obtain the substitution rule

$$S_{-+}: \quad A \mapsto A\bar{B}AB, \quad B \mapsto A\bar{B}\bar{A}\bar{B}, \quad \bar{A} \mapsto \bar{A}B\bar{A}\bar{B}, \quad \bar{B} \mapsto \bar{A}BAB. \tag{5.16}$$

This is a substitution of constant length four, because we used a double step of the recursion, and Equation (5.15) corresponds to concatenation of four sets of coefficients. As before, a one-sided fixed point sequence w_{-+} is obtained from iterating the substitution on the initial letter A ,

$$A \mapsto A\overline{B}AB \mapsto A\overline{B}AB\overline{A}BAB\overline{A}BAB\overline{A}BAB\overline{A}\overline{B} \mapsto \cdots \longrightarrow w_{-+}.$$

By mapping A, B to 1 and $\overline{A}, \overline{B}$ to $\overline{1} = -1$ using the map φ of Equation (5.6), we obtain the binary sequence $v_{-+} = \varphi(w_{-+}) = 1\overline{1}11\overline{1}111\overline{1}111\overline{1}\overline{1}\overline{1}\dots$ as our new RS-type sequence.

Alternatively, one can see the substitution S_{-+} as the composition of the two substitutions S_+ and S_- from Equations (5.5) and (5.11), in the sense that $S_{-+} = S_- \circ S_+$. To see this explicitly, let us verify the composition on the letters A and B ,

$$\begin{aligned} A &\xrightarrow{S_+} AB \xrightarrow{S_-} A\overline{B}AB, \\ B &\xrightarrow{S_+} A\overline{B} \xrightarrow{S_-} A\overline{B}\overline{A}\overline{B}. \end{aligned}$$

The images of \overline{A} and \overline{B} can be computed using the bar-swap symmetry.

From Proposition 5.3, we conclude that the binary sequence $v_{-+} = \varphi(w_{-+})$ satisfies the root- N property, and hence the corresponding spectral measure is absolutely continuous.

As previously mentioned, our construction does not satisfy the last condition of Theorem 5.1. The symbol matrix of the substitution S_{-+} of Equation (5.16) is given by

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix},$$

which is clearly not Hadamard.

We now employ Bartlett's algorithm from Section 3.6 to verify that the balanced weight version of the substitution system S_{-+} is absolutely continuous and in fact Lebesgue.

Proposition 5.5. *The balanced weight sequence S_{-+} has Lebesgue spectrum.*

Proof. The second iterate of the letter A is $A\bar{B}AB\bar{A}BAB\bar{A}BAB\bar{A}\bar{B}\bar{A}\bar{B}$, which shows that the letter B can be preceded by either A or \bar{A} . Hence the letter B has two distinct neighbourhoods, so, by Pansiot's lemma 2.12, the sequence is aperiodic.

The instruction matrices and the substitution matrix can be read off from the substitution rule of Equation (5.16) and are given by

$$R_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $M_{S_{-+}} = R_0 + R_1 + R_2 + R_3$. As $M_{S_{-+}}^2 > 0$, the substitution is primitive. We find the leading eigenvalue of the matrix $M_{S_{-+}}$ is 4 and $v_{\text{PF}} = \frac{1}{4}(1, 1, 1, 1)$ for the corresponding Perron–Frobenius vector.

To determine the extremal points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution $S_{-+} \otimes S_{-+}$.

$$E_1 : \begin{cases} AA \mapsto A A \bar{B} \bar{B} A A B B \\ B B \mapsto A A \bar{B} \bar{B} \bar{A} \bar{A} \bar{B} \bar{B} \\ \bar{B} \bar{B} \mapsto \bar{A} \bar{A} B B A A B B \\ \bar{A} \bar{A} \mapsto \bar{A} \bar{A} B B \bar{A} \bar{A} \bar{B} \bar{B} \end{cases} \quad E_2 : \begin{cases} A \bar{A} \mapsto A \bar{A} \bar{B} B A \bar{A} B \bar{B} \\ B \bar{B} \mapsto A \bar{A} \bar{B} B \bar{A} \bar{A} \bar{B} B \\ \bar{B} B \mapsto \bar{A} A B \bar{B} A \bar{A} B \bar{B} \\ \bar{A} A \mapsto \bar{A} A B \bar{B} \bar{A} \bar{A} \bar{B} B \end{cases}$$

and

$$T : \begin{cases} A B \mapsto A A \bar{B} \bar{B} A \bar{A} B \bar{B} \\ A \bar{B} \mapsto A \bar{A} \bar{B} B A A B B \\ B A \mapsto A A \bar{B} \bar{B} \bar{A} \bar{A} \bar{B} B \\ B \bar{A} \mapsto A \bar{A} \bar{B} B \bar{A} \bar{A} \bar{B} \bar{B} \\ \bar{B} A \mapsto \bar{A} A B \bar{B} A A B B \\ \bar{B} \bar{A} \mapsto \bar{A} \bar{A} B B A \bar{A} B \bar{B} \\ \bar{A} B \mapsto \bar{A} A B \bar{B} \bar{A} \bar{A} \bar{B} \bar{B} \\ \bar{A} \bar{B} \mapsto \bar{A} \bar{A} B B \bar{A} \bar{A} \bar{B} B \end{cases}$$

We have $E_1 = \{AA, BB, \bar{B}\bar{B}, \bar{A}\bar{A}\}$ and $E_2 = \{A\bar{A}, \bar{B}B, B\bar{B}, \bar{A}A\}$ as ergodic classes and the transient class $T = \{AB, A\bar{B}, B A, B \bar{A}, \bar{B} A, \bar{B} \bar{A}, \bar{A} B, \bar{A} \bar{B}\}$.

We then apply Proposition 3.5, with

$$\mathbf{v} = \begin{pmatrix} 1 & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & w \\ \frac{1}{2}(1+w) & 1 & w & \frac{1}{2}(1+w) \\ \frac{1}{2}(1+w) & w & 1 & \frac{1}{2}(1+w) \\ w & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & 1 \end{pmatrix} \geq 0. \quad (5.17)$$

We then diagonalise the matrix \mathbf{v} and obtain

$$\mathbf{v}_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-w & 0 & 0 \\ 0 & 0 & 1-w & 0 \\ 0 & 0 & 0 & 2+2w \end{pmatrix},$$

so v is strongly semi-positive if and only if $-1 \leq w \leq 1$. Hence $v \in \mathcal{K}(S)$ if and only if \mathbf{v} is of the form of Equation (5.17) with the condition $-1 \leq w \leq 1$. \mathcal{K}^* is given by the vectors

$$v_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

$$v_2 = (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1).$$

As the PF vector is $\frac{1}{4}(1, 1, 1, 1)$ so $\widehat{\Sigma}(0) = \frac{1}{4} \sum_{\gamma \in \mathcal{A}} e_{\gamma\gamma}$. Since this is a length four substitution, $q = 4$ and $\Delta_1(4) = \{3\}$. Equation (3.4) gives us

$$\begin{aligned} \widehat{\Sigma}(1) &= (4\mathbf{I} - R_3 \otimes R_0)^{-1} (R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(0). \\ &= \frac{1}{8} (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0). \end{aligned}$$

Using Theorem 3.3 for $k = 2$ and $p = 1$,

$$\begin{aligned} \widehat{\Sigma}(2) &= \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(0) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(1) \right) \\ &= \frac{1}{8} (1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1). \end{aligned}$$

Observe that $\widehat{\Sigma}(1)$ is orthogonal to $\widehat{\Sigma}(2)$.

Using Theorem 3.3, we obtain

$$\begin{aligned} \widehat{\Sigma}(3) &= \frac{1}{4} \sum_{j \in [0,4)} (R_j \otimes R_{j+3}) \widehat{\Sigma}(\lfloor j+3 \rfloor) \\ &= \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(0) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(1) \right) \\ &= \frac{1}{8} (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0) = \widehat{\Sigma}(1) \end{aligned}$$

and

$$\begin{aligned}
\widehat{\Sigma}(4) &= \frac{1}{4} \sum_{j \in [0,4)} (R_j \otimes R_{j+4}) \widehat{\Sigma}(\lfloor j+4 \rfloor) \\
&= \frac{1}{4} \left((R_0 \otimes R_0 + R_1 \otimes R_1 + R_2 \otimes R_2 + R_3 \otimes R_3) \widehat{\Sigma}(1) \right) \\
&= \frac{1}{4} C_{S_{-+}} \widehat{\Sigma}(1) \\
&= \frac{1}{8} (1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1) = \widehat{\Sigma}(2).
\end{aligned}$$

By direct computation and the Fourier recursion theorem 3.3, one obtains the following relations.

$$\begin{aligned}
&\left\{ \begin{aligned} \widehat{\Sigma}(4n) &= \frac{1}{4} (R_0 \otimes R_0 + R_1 \otimes R_1 + R_2 \otimes R_2 + R_3 \otimes R_3) \widehat{\Sigma}(n) = \frac{1}{4} C_{S_{-+}} \widehat{\Sigma}(n), \\ \frac{1}{4} C_{S_{-+}} \widehat{\Sigma}(1) &= \frac{1}{4} C_{S_{-+}} \widehat{\Sigma}(2) = \frac{1}{4} C_{S_{-+}} \widehat{\Sigma}(3) = \frac{1}{4} C_{S_{-+}} \widehat{\Sigma}(4) = \widehat{\Sigma}(4) = \widehat{\Sigma}(2), \end{aligned} \right. \\
&\left\{ \begin{aligned} \widehat{\Sigma}(4n+1) &= \frac{1}{4} \left((R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(n) + \frac{1}{4} (R_3 \otimes R_0) \widehat{\Sigma}(n+1) \right), \\ \frac{1}{4} \left((R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(1) + \frac{1}{4} (R_3 \otimes R_0) \widehat{\Sigma}(2) \right) &= \widehat{\Sigma}(1), \\ \frac{1}{4} \left((R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(2) + \frac{1}{4} (R_3 \otimes R_0) \widehat{\Sigma}(1) \right) &= \widehat{\Sigma}(1), \end{aligned} \right. \\
&\left\{ \begin{aligned} \widehat{\Sigma}(4n+2) &= \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(n) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(n+1) \right), \\ \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(1) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(2) \right) &= \widehat{\Sigma}(2), \\ \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(2) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(1) \right) &= \widehat{\Sigma}(2), \end{aligned} \right. \\
&\left\{ \begin{aligned} \widehat{\Sigma}(4n+3) &= \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(n) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(n+1) \right), \\ \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(1) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(2) \right) &= \widehat{\Sigma}(1), \\ \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(2) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(1) \right) &= \widehat{\Sigma}(1). \end{aligned} \right.
\end{aligned}$$

We then proceed by induction and obtain the identities $\widehat{\Sigma}(4n) = \widehat{\Sigma}(2)$, $\widehat{\Sigma}(4n+1) = \widehat{\Sigma}(1)$, $\widehat{\Sigma}(4n+2) = \widehat{\Sigma}(2)$ and $\widehat{\Sigma}(4n+3) = \widehat{\Sigma}(1)$ for $n \neq 0$.

As usual, $\lambda_{v_1} = \delta_0$. Using the Fourier coefficients and identities we obtain, $\widehat{\lambda_{v_2}}(4n) = v_2^t \widehat{\Sigma}(2) = 0$, $\widehat{\lambda_{v_2}}(4n+1) = v_2^t \widehat{\Sigma}(1) = 0$, $\widehat{\lambda_{v_2}}(4n+2) = v_2^t \widehat{\Sigma}(2) = 0$ and $\widehat{\lambda_{v_2}}(4n+3) = v_2^t \widehat{\Sigma}(1) = 0$, so λ_{v_2} is Lebesgue measure. Therefore the balanced weight sequence S_{-+} has Lebesgue spectrum. \square

Our next example is closely related. We again alternate the signs in the recursion, but shifted by one. Maybe surprisingly, this produces a different sequence of coefficients.

Example 5.3.4. Here we choose $\sigma_k = (-1)^k$. The recurrence relations are now

$$\begin{aligned} P_{k+1}(x) &= P_k(x) + (-1)^k x^{2^k} Q_k(x), \\ Q_{k+1}(x) &= P_k(x) - (-1)^k x^{2^k} Q_k(x), \end{aligned} \quad (5.18)$$

for $k \in \mathbb{N}_0$. Using the same approach as above, this gives rise to the substitution rule

$$S_{+-}: \quad A \mapsto AB\bar{A}B, \quad B \mapsto ABAB\bar{B}, \quad \bar{A} \mapsto \bar{A}\bar{B}A\bar{B}, \quad \bar{B} \mapsto \bar{A}\bar{B}\bar{A}B. \quad (5.19)$$

This rule can also be expressed as the composition of the two substitution systems S_+ and S_- , this time as $S_{+-} = S_+ \circ S_-$, because

$$\begin{aligned} A &\xrightarrow{S_-} A\bar{B} \xrightarrow{S_+} AB\bar{A}B, \\ B &\xrightarrow{S_-} AB \xrightarrow{S_+} ABAB\bar{B}, \end{aligned}$$

and the relations for the barred letters follow by bar-swap symmetry. Again, Proposition 5.3 shows that the corresponding binary sequence $v_{+-} = \varphi(w_{+-})$, where w_{+-} denotes the fixed point of S_{+-} obtained by iterating S_{+-} on the letter A , satisfies the root- N property and hence the spectral measure for the binary sequence v_{+-} is absolutely continuous.

Once again, we can write down the symbol matrix of the substitution S_{+-} of Equation (5.19) and show it does not satisfy the last condition of Theorem 5.1 and it is given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix},$$

which is clearly not Hadamard.

We now employ Bartlett's algorithm from Section 3.6 to verify the balanced weight version of the substitution system S_{+-} is absolutely continuous and in fact Lebesgue.

Proposition 5.6. *The balanced weight sequence S_{+-} has Lebesgue spectrum.*

Proof. The second iterate of the letter A is $AB\bar{A}BABAB\bar{B}\bar{A}\bar{B}A\bar{B}ABAB\bar{B}$, which shows that the letter B can be preceded by both A or \bar{A} . Hence the

letter B has two distinct neighbourhoods, so, by Pansiot's lemma 2.12, the sequence is aperiodic.

The instruction matrices and the substitution matrix can be read off from the substitution rule of Equation (5.19) and are given by

$$R_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $M_{S_{+-}} = R_0 + R_1 + R_2 + R_3$. As $M_{S_{+-}}^2 > 0$, the substitution is primitive. We find the leading eigenvalue of the matrix $M_{S_{+-}}$ is 4 and $v_{\text{PF}} = \frac{1}{4}(1, 1, 1, 1)$ for the corresponding Perron–Frobenius vector.

To determine the extremal points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution $S_{+-} \otimes S_{+-}$.

$$E_1 : \begin{cases} AA \mapsto AABBA\bar{A}AB\bar{B} \\ BB \mapsto AABBA\bar{A}\bar{B}\bar{B} \\ \bar{B}\bar{B} \mapsto \bar{A}\bar{A}\bar{B}\bar{B}\bar{A}\bar{A}BB \\ \bar{A}\bar{A} \mapsto \bar{A}\bar{A}\bar{B}\bar{B}AA\bar{B}\bar{B} \end{cases} \quad E_2 : \begin{cases} A\bar{A} \mapsto A\bar{A}B\bar{B}\bar{A}AB\bar{B} \\ B\bar{B} \mapsto A\bar{A}B\bar{B}A\bar{A}\bar{B}B \\ \bar{B}B \mapsto \bar{A}\bar{A}\bar{B}B\bar{A}AB\bar{B} \\ \bar{A}A \mapsto \bar{A}\bar{A}\bar{B}B\bar{A}\bar{B}B \end{cases}$$

and

$$T : \begin{cases} AB \mapsto AABBA\bar{A}AB\bar{B} \\ A\bar{B} \mapsto A\bar{A}B\bar{B}\bar{A}\bar{A}BB \\ BA \mapsto AABBA\bar{A}\bar{B}B \\ B\bar{A} \mapsto A\bar{A}B\bar{B}A\bar{A}\bar{B}\bar{B} \\ \bar{B}A \mapsto \bar{A}\bar{A}\bar{B}B\bar{A}\bar{A}BB \\ \bar{B}\bar{A} \mapsto \bar{A}\bar{A}\bar{B}\bar{B}\bar{A}AB\bar{B} \\ \bar{A}B \mapsto \bar{A}\bar{A}\bar{B}B\bar{A}\bar{B}\bar{B} \\ \bar{A}\bar{B} \mapsto \bar{A}\bar{A}\bar{B}\bar{B}A\bar{A}\bar{B}B \end{cases}$$

We have $E_1 = \{AA, BB, \bar{B}\bar{B}, \bar{A}\bar{A}\}$ and $E_2 = \{A\bar{A}, \bar{B}B, B\bar{B}, \bar{A}A\}$ as ergodic classes and the transient class $T = \{A\bar{B}, A\bar{B}, B\bar{A}, B\bar{A}, \bar{B}A, \bar{B}A, \bar{A}B, \bar{A}B\}$.

We then apply Proposition 3.5, with

$$\mathbf{v} = \begin{pmatrix} 1 & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & w \\ \frac{1}{2}(1+w) & 1 & w & \frac{1}{2}(1+w) \\ \frac{1}{2}(1+w) & w & 1 & \frac{1}{2}(1+w) \\ w & \frac{1}{2}(1+w) & \frac{1}{2}(1+w) & 1 \end{pmatrix} \geq 0. \quad (5.20)$$

We then diagonalise the matrix \mathbf{v} and obtain

$$\mathbf{v}_d = \begin{pmatrix} 2+2w & 0 & 0 & 0 \\ 0 & 1-w & 0 & 0 \\ 0 & 0 & 1-w & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so v is strongly semi-positive if and only if $-1 \leq w \leq 1$. Hence $v \in \mathcal{K}(S)$ if and only if \mathbf{v} is of the form of Equation (5.20) with the condition $-1 \leq w \leq 1$. \mathcal{K}^* is given by the vectors

$$v_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

$$v_2 = (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 1, 0, -1, 0, 0, 1).$$

As the PF vector is $\frac{1}{4}(1, 1, 1, 1)$ so $\widehat{\Sigma}(0) = \frac{1}{4} \sum_{\gamma \in \mathcal{A}} e_{\gamma\gamma}$. Since this is a length four substitution, $q = 4$ and $\Delta_1(4) = \{3\}$. Equation (3.4) gives us

$$\begin{aligned} \widehat{\Sigma}(1) &= (4\mathbf{I} - R_3 \otimes R_0)^{-1}(R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(0). \\ &= \frac{1}{8}(0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0). \end{aligned}$$

Using Theorem 3.3 for $k = 2$ and $p = 1$,

$$\begin{aligned} \widehat{\Sigma}(2) &= \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(0) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(1) \right) \\ &= \frac{1}{8}(1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1). \end{aligned}$$

Observe that $\widehat{\Sigma}(1)$ is orthogonal to $\widehat{\Sigma}(2)$. Using Theorem 3.3, we obtain

$$\begin{aligned} \widehat{\Sigma}(3) &= \frac{1}{4} \sum_{j \in [0,4)} (R_j \otimes R_{j+3}) \widehat{\Sigma}(\lfloor j+3 \rfloor) \\ &= \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(0) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(1) \right) \\ &= \frac{1}{8}(0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0) = \widehat{\Sigma}(1) \end{aligned}$$

and

$$\begin{aligned} \widehat{\Sigma}(4) &= \frac{1}{4} \sum_{j \in [0,4)} (R_j \otimes R_{j+4}) \widehat{\Sigma}(\lfloor j+4 \rfloor) \\ &= \frac{1}{4} \left((R_0 \otimes R_0 + R_1 \otimes R_1 + R_2 \otimes R_2 + R_3 \otimes R_3) \widehat{\Sigma}(1) \right) \\ &= \frac{1}{4} C_{S-+} \widehat{\Sigma}(1) \\ &= \frac{1}{8}(1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1) = \widehat{\Sigma}(2). \end{aligned}$$

By direct computation and the Fourier recursion theorem 3.3, one obtains the following relations.

$$\begin{cases} \widehat{\Sigma}(4n) = \frac{1}{4} (R_0 \otimes R_0 + R_1 \otimes R_1 + R_2 \otimes R_2 + R_3 \otimes R_3) \widehat{\Sigma}(n) = \frac{1}{4} C_{S_{+-}} \widehat{\Sigma}(n), \\ \frac{1}{4} C_{S_{+-}} \widehat{\Sigma}(1) = \frac{1}{4} C_{S_{+-}} \widehat{\Sigma}(2) = \frac{1}{4} C_{S_{+-}} \widehat{\Sigma}(3) = \frac{1}{4} C_{S_{+-}} \widehat{\Sigma}(4) = \widehat{\Sigma}(2), \end{cases}$$

$$\begin{cases} \widehat{\Sigma}(4n+1) = \frac{1}{4} \left((R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(n) + \frac{1}{4} (R_3 \otimes R_0) \widehat{\Sigma}(n+1) \right), \\ \frac{1}{4} \left((R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(1) + \frac{1}{4} (R_3 \otimes R_0) \widehat{\Sigma}(2) \right) = \widehat{\Sigma}(1), \\ \frac{1}{4} \left((R_0 \otimes R_1 + R_1 \otimes R_2 + R_2 \otimes R_3) \widehat{\Sigma}(2) + \frac{1}{4} (R_3 \otimes R_0) \widehat{\Sigma}(1) \right) = \widehat{\Sigma}(1), \end{cases}$$

$$\begin{cases} \widehat{\Sigma}(4n+2) = \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(n) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(n+1) \right), \\ \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(1) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(2) \right) = \widehat{\Sigma}(2), \\ \frac{1}{4} \left((R_0 \otimes R_2 + R_1 \otimes R_3) \widehat{\Sigma}(2) + (R_2 \otimes R_0 + R_3 \otimes R_1) \widehat{\Sigma}(1) \right) = \widehat{\Sigma}(2), \end{cases}$$

$$\begin{cases} \widehat{\Sigma}(4n+3) = \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(n) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(n+1) \right), \\ \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(1) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(2) \right) = \widehat{\Sigma}(1), \\ \frac{1}{4} \left((R_0 \otimes R_3) \widehat{\Sigma}(2) + (R_1 \otimes R_0 + R_2 \otimes R_1 + R_3 \otimes R_2) \widehat{\Sigma}(1) \right) = \widehat{\Sigma}(1). \end{cases}$$

We then proceed by induction and obtain the identities $\widehat{\Sigma}(4n) = \widehat{\Sigma}(2)$, $\widehat{\Sigma}(4n+1) = \widehat{\Sigma}(1)$, $\widehat{\Sigma}(4n+2) = \widehat{\Sigma}(2)$ and $\widehat{\Sigma}(4n+3) = \widehat{\Sigma}(1)$ for $n \neq 0$.

As usual, $\lambda_{v_1} = \delta_0$. Using the Fourier coefficients and identities we obtain, $\widehat{\lambda_{v_2}}(4k) = v_2^t \widehat{\Sigma}(2) = 0$, $\widehat{\lambda_{v_2}}(4k+1) = v_2^t \widehat{\Sigma}(1) = 0$, $\widehat{\lambda_{v_2}}(4k+2) = v_2^t \widehat{\Sigma}(2) = 0$ and $\widehat{\lambda_{v_2}}(4k+3) = v_2^t \widehat{\Sigma}(1) = 0$, so λ_{v_2} is Lebesgue measure. Therefore the balanced weight sequence S_{+-} has Lebesgue spectrum. \square

The two substitutions S_{-+} and S_{+-} and their respective hulls are closely related.

Lemma 5.7. *The hulls \mathbb{X}_{-+} and \mathbb{X}_{+-} of the substitutions S_{-+} and S_{+-} defined by Equations (5.16) and (5.19) satisfy the relations*

$$\begin{aligned} \mathbb{X}_{+-} &= S_+(\mathbb{X}_{-+}) \cup TS_+(\mathbb{X}_{-+}), \\ \mathbb{X}_{-+} &= S_-(\mathbb{X}_{+-}) \cup TS_-(\mathbb{X}_{+-}), \end{aligned}$$

where T denotes the shift map on $\{A, \overline{A}, B, \overline{B}\}^{\mathbb{N}}$.

Proof. If w_{-+} and w_{+-} denote fixed point sequences for S_{-+} and S_{+-} , then the (one-sided) hulls \mathbb{X}_{-+} and \mathbb{X}_{+-} are the closures of orbits of the fixed points under the shift map T . Now, $S_{-+} = S_- \circ S_+$ implies that

$$S_{-+}(S_-w_{+-}) = (S_- \circ S_+ \circ S_-)w_{+-} = S_-(S_{+-}w_{+-}) = S_-w_{+-},$$

which shows that S_-w_{+-} is a fixed point of S_{-+} . Similarly, since $S_{+-} = S_+ \circ S_-$, we have

$$S_{+-}(S_+w_{-+}) = (S_+ \circ S_- \circ S_+)w_{-+} = S_+(S_{-+}w_{-+}) = S_+w_{-+},$$

and consequently S_+w_{-+} is a fixed point of S_{+-} .

Since the substitutions S_+ and S_- have constant length two, we have

$$S_+ \circ T = T^2 \circ S_+ \quad \text{and} \quad S_- \circ T = T^2 \circ S_-, \quad (5.21)$$

which implies that $S_+(\mathbb{X}_{-+})$ is the subset of \mathbb{X}_{+-} of all sequences starting with a letter A or \bar{A} , since only even shifts are included. By continuity of the action, limits are included, so the closure does not add any additional elements. Hence the union $S_+(\mathbb{X}_{-+}) \cup TS_+(\mathbb{X}_{-+})$ gives the complete hull \mathbb{X}_{+-} , and the analogous result holds for the case where the signs are interchanged. \square

Notice that, despite the above connection, the hulls \mathbb{X}_{-+} and \mathbb{X}_{+-} are indeed different. This is a consequence of Proposition 2.4, as there are words of length six that appears in one hull but not the other.

Despite the hulls \mathbb{X}_{-+} and \mathbb{X}_{+-} being different, the two systems are still intimately connected. We recall the following definition from [25, Sect. 2.2]

Definition 5.3.5. Let (\mathbb{X}, T) be a minimal dynamical system, and $U \subset \mathbb{X}$ a clopen set. Let $T_U : U \rightarrow U$ be the *induced transformation*, i.e. if $x \in U$ then

$$T_U(x) = T^{r_U(x)}(x), \quad \text{where } r_U(x) = \inf\{n > 0 : T^n(x) \in U\}.$$

The pair (U, T_U) is called *induced system of (\mathbb{X}, T) with respect to U* and $r_U(x)$ is called the *return time function*.

Proposition 5.8. (\mathbb{X}_{-+}, T) is conjugate to the induced system of (\mathbb{X}_{+-}, T) on the subset $S_+(\mathbb{X}_{-+})$, and (\mathbb{X}_{+-}, T) is conjugate to the induced system of (\mathbb{X}_{-+}, T) on the subset $S_-(\mathbb{X}_{+-})$.

Proof. Here we prove the first claim; the second follows analogously. As mentioned in the proof of Lemma 5.7, $S_+(\mathbb{X}_{-+}) = [A] \cup [\bar{A}] \subset \mathbb{X}_{+-}$, where the brackets denote cylinder sets of words starting with the given letter. Now, consider the return time function of Definition 5.3.5, that is, the return time of the fixed point generated by S_{+-} to the clopen set $[A] \cup [\bar{A}]$,

$$r_{[A] \cup [\bar{A}]} = \inf\{n > 0 : T^n(w_{+-}) \in [A] \cup [\bar{A}]\}.$$

As S_+ is a substitution of length two, each letter is mapped into a length two word starting with A or \bar{A} and it follows that $r_{[A] \cup [\bar{A}]} = 2$. The induced map is then given by T^2 , which maps the set $[A] \cup [\bar{A}]$ onto itself. Hence, $([A] \cup [\bar{A}], T^2)$ is the induced system. Using the observation we made in the first part of Equation (5.21), namely $S_+ \circ T = T^2 \circ S_+$, we just have to prove that the map $S_+ : \mathbb{X}_{-+} \rightarrow S_+(\mathbb{X}_{-+})$ is injective, which is obvious from the fact that S_+ is an injective substitution. Therefore the conjugacy follows. \square

By using the following result [49, Thm. 2.9], the induced systems inherit the spectral properties of the conjugated systems.

Theorem 5.9. *Let T_i with $i \in \{1, 2\}$ be measure-preserving transformations of probability spaces. If T_1 and T_2 are conjugate, then they are spectrally isomorphic.*

Note that, as previously, the four-letter hull \mathbb{X}_{-+} and two-letter hull $\varphi(\mathbb{X}_{-+})$ are mutually locally derivable (as are \mathbb{X}_{+-} and $\varphi(\mathbb{X}_{+-})$), and the corresponding dynamical systems are hence topologically conjugate. The argument is the same as above; the subword 1111, which occurs in both $\varphi(w_{-+})$ and $\varphi(w_{+-})$ with bounded gaps, has the unique preimage $BABA$ in both w_{-+} and w_{+-} .

The observations of Examples 5.3.3 and 5.3.4 suggest the following general picture.

Theorem 5.10. *Let $(\sigma_k)_{k \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}_0}$ be a given sequence. Then, for any $k \in \mathbb{N}_0$, the sequence of coefficients of the polynomial P_k defined by Equation (5.8) is the image under the map φ of $A_k = S_{\sigma_0} \circ S_{\sigma_1} \circ \cdots \circ S_{\sigma_{k-1}} A$.*

Proof. Let $a_k := \varepsilon_1 \varepsilon_2 \dots \varepsilon_{2^k} \in \{\pm 1\}^{2^k}$ denote the word of length 2^k of coefficients of $P_k(x)$, and b_k the corresponding word for $Q_k(x)$. Then the recurrence

relations of Equation (5.8) correspond to the concatenation relations

$$a_{k+1} = \begin{cases} a_k b_k, & \sigma_k = 1, \\ a_k \overline{b_k}, & \sigma_k = -1, \end{cases} \quad b_{k+1} = \begin{cases} a_k \overline{b_k}, & \sigma_k = 1, \\ a_k b_k, & \sigma_k = -1, \end{cases}$$

with initial values $a_0 = b_0 = 1$. These recurrence relations correspond to the substitution rule S_{σ_k} , and by induction we obtain $a_k = \varphi(A_k)$ with

$$A_k = S_{\sigma_0} \circ \cdots \circ S_{\sigma_{k-1}} A$$

for any $k \in \mathbb{N}_0$. □

Clearly, if we choose $\sigma_k = 1$ for all $k \in \mathbb{N}_0$, we are back at the RS case with substitution S_+ . More generally, for any periodic sequence we have the following result.

Corollary 5.11. *Let $(\sigma_k)_{k \in \mathbb{N}_0} \in \{\pm 1\}^{\mathbb{N}_0}$ be a periodic sequence of period p , so $\sigma_{k+p} = \sigma_k$ for all $k \in \mathbb{N}_0$. Then, the sequence of coefficients of the polynomials P_k defined by Equation (5.8) is the image under the map φ of the fixed point of the substitution*

$$S_{\sigma_0 \sigma_1 \dots \sigma_{p-1}} := S_{\sigma_0} \circ S_{\sigma_1} \circ \cdots \circ S_{\sigma_{p-1}}$$

with initial letter A .

Proof. As the sequence of signs σ_k is periodic with period p , Theorem 5.10 implies that

$$A_{np} = (S_{\sigma_0} \circ \cdots \circ S_{\sigma_{p-1}})^n A$$

holds for $n \in \mathbb{N}$, and the assertion follows. □

Combining Corollary 5.11 and the concept of MLD. We have the following result.

Proposition 5.12. *Let $(\sigma_k)_{k \in \mathbb{N}} \in \{\pm 1\}^{\mathbb{N}}$ be a periodic sequence of period p and $S_{\sigma_1 \sigma_2 \dots \sigma_p}$ be the corresponding substitution according to Corollary 5.11. Its hull $\mathbb{X}_{\sigma_1 \sigma_2 \dots \sigma_p}$ is then mutually locally derivable with $\varphi(\mathbb{X}_{\sigma_1 \sigma_2 \dots \sigma_p})$.*

Proof. Local derivability of the two-letter sequence from the four-letter sequence is clear, as φ acts locally.

To show local derivability of the four-letter sequence, note that $BABA$ is a legal four-letter word for S_+ and S_- as well as for $S_{-+} = S_- \circ S_+$ and $S_{+-} = S_+ \circ S_-$. Hence, it is also legal for $S_{\sigma_1\sigma_2\cdots\sigma_p} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_p}$, and occurs with bounded gaps in any element of the hull $\mathbb{X}_{\sigma_1\sigma_2\cdots\sigma_p}$ by repetitivity of the hull. Since $\varphi(BABA) = 1111$, the latter also occurs with bounded gaps in any element of the two-letter hull.

Now, observe that $ABAB$ is not a legal word for S_+ or S_- (as its pre-image would have to be AA or BB), or for $S_{-+} = S_- \circ S_+$ or $S_{+-} = S_+ \circ S_-$. As a consequence, it cannot occur as a legal word for $S_{\sigma_1\sigma_2\cdots\sigma_p} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_p}$ either.

Hence $BABA$ is the unique pre-image of 1111, and local derivability follows. \square

As mentioned previously, when the two substitution systems are MLD, the map φ is a bijection and as it commutes with the shift action, it gives rise to topological conjugacy. We can then invoke Theorem 5.9 [49] and that tells us that once we know the spectral properties of the binary hull, the quaternary hull will have the same spectral properties. Therefore, Proposition 5.12 implies both the two-letter and four-letter hulls from the root- N construction have absolutely continuous/Lebesgue spectrum.

6

GENERALISATION OF RUDIN'S METHOD: COMPLEX WEIGHTS

Contents

6.1	Generalisation to complex weights	76
6.2	Examples	79

6.1 Generalisation to complex weights

In Chapter 5, we considered polynomials with real coefficients in the recurrence relations (5.8) and showed that they satisfy the root- N property of Equation (5.1). In this chapter, we are going to generalise and extend the argument of Rudin further by considering *complex* coefficients in our polynomials. This will naturally lead us to look at the following types of matrices.

Definition 6.1.1. A *Fourier matrix* of order n is a unitary $n \times n$ matrix with the (j, k) -entry is $\frac{1}{\sqrt{n}} \exp\left(\frac{2\pi i(j-1)(k-1)}{n}\right)$, where $1 \leq j, k \leq n$.

We refer the reader to [48] for further details and recent developments on the theory of Fourier matrices.

The matrices that we are going to consider from now on will be x -dependent generalisations of these Fourier matrices, without the normalisation factor $1/\sqrt{n}$, where x is the variable in the recursion polynomials.

It will be convenient to express the recurrence relations of Equation (5.2) in terms of matrices as follows,

$$\begin{pmatrix} P_{k+1}(x) \\ Q_{k+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x^{2^k} \\ 1 & -x^{2^k} \end{pmatrix} \begin{pmatrix} P_k(x) \\ Q_k(x) \end{pmatrix} = A^{(2,k)} \begin{pmatrix} P_k(x) \\ Q_k(x) \end{pmatrix}.$$

Now, for $n > 2$, consider a vector of n polynomials

$$z_k = \begin{pmatrix} P_k^{(1)}(x) \\ \vdots \\ P_k^{(n)}(x) \end{pmatrix}$$

satisfying the recurrence relation

$$z_{k+1} = A^{(n,k)} z_k, \quad (6.1)$$

with initial condition $z_0 = (x, \dots, x)^t$. Here, $A^{(n,k)}$ is the $n \times n$ matrix

$$A^{(n,k)} = \begin{pmatrix} 1 & x^{n^k} & \dots & x^{(n-1)n^k} \\ 1 & \omega x^{n^k} & \dots & \omega^{n-1} x^{(n-1)n^k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} x^{n^k} & \dots & \omega^{(n-1)^2} x^{(n-1)n^k} \end{pmatrix},$$

where $\omega = \exp(2\pi i/n)$. For $x = 1$, $A^{(n,k)}$ reduces to the $n \times n$ Fourier matrix, apart from the normalisation factor $1/\sqrt{n}$. As a consequence, for $|x| = 1$, the matrix satisfies $(A^{(n,k)})^\dagger A^{(n,k)} = n \mathbf{1}^{(n)}$, where $\mathbf{1}^{(n)}$ denotes the $n \times n$ identity matrix and $M^\dagger = \overline{M}^t$ denotes the Hermitian adjoint of the matrix (or vector) M .

Generalising Equation (5.3), we can now define a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of complex coefficients $\varepsilon_m \in \{\omega^j : 0 \leq j \leq n-1\}$ by

$$P_k^{(1)}(x) = \sum_{m=1}^{n^k} \varepsilon_m x^m. \quad (6.2)$$

We shall show that these sequences also satisfy the root- N property of Equation (5.1). To do this, we will need the unitary property of Fourier matrices.

Theorem 6.1. *The sequence of coefficients $(\varepsilon_m)_{m \in \mathbb{N}}$ of the functions $P_k^{(1)}(x)$ satisfies the root- N property of Equation (5.1).*

Proof. The proof proceeds by induction. We want to derive a bound for $|P_{k+1}^{(1)}(x)|$. To do so, we express the sum of the squared norms of the polynomials $P_{k+1}^{(1)}(x), \dots, P_{k+1}^{(n)}(x)$ as

$$z_{k+1}^\dagger z_{k+1} = \sum_{j=1}^n |P_{k+1}^{(j)}(x)|^2.$$

Using the recurrence relation of Equation (6.1) and the identity $(A^{(n,k)})^\dagger A^{(n,k)} = n \mathbf{1}^{(n)}$, we obtain

$$z_{k+1}^\dagger z_{k+1} = z_k^\dagger (A^{(n,k)})^\dagger A^{(n,k)} z_k = n z_k^\dagger z_k = n \left(\sum_{j=1}^n |P_k^{(j)}(x)|^2 \right).$$

This shows that

$$\sum_{j=1}^n |P_{k+1}^{(j)}(x)|^2 = n \left(\sum_{j=1}^n |P_k^{(j)}(x)|^2 \right).$$

Since we have $\sum_{j=1}^n |P_0^{(j)}(x)|^2 = n$ by the initial conditions, we conclude by induction that

$$\sum_{j=1}^n |P_k^{(j)}(x)|^2 = n^{k+1}.$$

Hence we get the bound

$$|P_k^{(1)}(x)| \leq n^{\frac{k+1}{2}}, \quad (6.3)$$

and in particular $|P_k^{(1)}(x)| \leq n^{\frac{1}{2}} n^{\frac{k}{2}}$, which proves the root- N property for $N = n^k$.

It remains to prove the property for other values of N . The argument is similar to that used in the proof of Proposition 5.3. Let $P_{k|m}^{(j)}$ denote the m -th partial sum of $P_k^{(j)}$ for $1 \leq j \leq n$, where $n^{k-1} < m \leq n^k$. We will prove by induction that these functions satisfy

$$|P_{k|m}^{(j)}(x)| \leq G n^{\frac{k}{2}} \quad (6.4)$$

for all $|x| = 1$ and $k \in \mathbb{N}_0$, where $G = n + n^{\frac{1}{2}}$.

Clearly, this estimate is true if $k = 0$. Suppose now that Equation (6.4) holds for some $k \in \mathbb{N}_0$, and consider an integer m with $n^k < m \leq n^{k+1}$. For $n^k < m \leq 2n^k$, by using the recursion (6.1) as well as the triangle inequality together with Equations (6.3) and (6.4), we obtain

$$|P_{k+1|m}^{(j)}(x)| \leq |P_k^{(1)}(x)| + |\omega^{j-1} x^{n^k} P_{k|m-n^k}^{(2)}(x)| \leq n^{\frac{k+1}{2}} + G n^{\frac{k}{2}} \leq G n^{\frac{k+1}{2}}$$

for all $1 \leq j \leq n$.

Similarly, we can derive bounds for the cases where $\ell n^k < m \leq (\ell + 1)n^k$ for all $1 \leq \ell \leq n - 1$, where more and more terms contribute. We obtain

$$\begin{aligned} |P_{k+1|m}^{(j)}(x)| &= \left| \sum_{r=1}^{\ell} \omega^{(r-1)(j-1)} x^{(r-1)n^k} P_k^{(r)}(x) + \omega^{\ell(j-1)} x^{\ell n^k} P_{k|m-\ell n^k}^{(\ell+1)}(x) \right| \\ &\leq \sum_{r=1}^{\ell} |\omega^{(r-1)(j-1)} x^{(r-1)n^k} P_k^{(r)}(x)| + |\omega^{\ell(j-1)} x^{\ell n^k} P_{k|m-\ell n^k}^{(\ell+1)}(x)| \\ &\leq \sum_{r=1}^{\ell} |P_k^{(r)}(x)| + |P_{k|m-\ell n^k}^{(\ell+1)}(x)| \\ &\leq \ell n^{\frac{k+1}{2}} + G n^{\frac{k}{2}} \\ &\leq ((n-1) + (n^{\frac{1}{2}} + 1)) n^{\frac{k+1}{2}} = G n^{\frac{k+1}{2}}, \end{aligned}$$

which completes the induction argument.

To finish the proof, suppose that $n^{k-1} < N \leq n^k$. By Equation (6.4), we have

$$|P_{k|N}^{(1)}(x)| \leq (n + n^{\frac{1}{2}}) n^{\frac{k}{2}} \leq n(n^{\frac{1}{2}} + 1) N^{\frac{1}{2}},$$

which shows that the root- N property holds. \square

Note that the case $n = 2$ corresponds to Equation (5.2), which is the RS case. Let us now consider some examples.

6.2

Examples

Example 6.2.1. Consider the case when $n = 3$. We start by defining three sequences of polynomials, $(P_k(x))_{k \in \mathbb{N}_0}$, $(Q_k(x))_{k \in \mathbb{N}_0}$ and $(R_k(x))_{k \in \mathbb{N}_0}$, where P_k , Q_k and R_k all have degree 3^k . They are determined by the initial choices $P_0(x) = Q_0(x) = R_0(x) = x$ together with the recurrence relations

$$\begin{pmatrix} P_{k+1}(x) \\ Q_{k+1}(x) \\ R_{k+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x^{3^k} & x^{2 \cdot 3^k} \\ 1 & \omega x^{3^k} & \omega^2 x^{2 \cdot 3^k} \\ 1 & \omega^2 x^{3^k} & \omega x^{2 \cdot 3^k} \end{pmatrix} \begin{pmatrix} P_k(x) \\ Q_k(x) \\ R_k(x) \end{pmatrix}, \quad (6.5)$$

where $k \in \mathbb{N}_0$ and $\omega = \exp(\frac{2\pi i}{3})$. By construction,

$$P_k(x) = \sum_{n=1}^{3^k} \varepsilon_n x^n,$$

where each coefficient ε_n is either 1 or ω or ω^2 , so we can define a ternary sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in \{1, \omega, \omega^2\}^{\mathbb{N}}$ from the coefficients. From here (and henceforth) we use the convention of a single bar to describe multiplication by ω and double bar to describe multiplication by ω^2 . If $a_k = \varepsilon_1 \cdots \varepsilon_{3^k} \in \{1, \omega, \omega^2\}^{3^k}$ denotes the word of length 3^k of coefficients of $P_k(x)$, b_k denotes the corresponding word for $Q_k(x)$ and c_k denotes the corresponding word for $R_k(x)$, then the recurrence relations of Equation (6.5) correspond to the concatenation relations

$$\begin{aligned} a_{k+1} &= a_k b_k c_k, \\ b_{k+1} &= a_k \overline{b_k} \overline{c_k}, \\ c_{k+1} &= a_k \overline{\overline{b_k}} \overline{\overline{c_k}}, \end{aligned} \quad (6.6)$$

on words in the three-letter alphabet $\{1, \omega, \omega^2\}$, with initial values $a_0 = b_0 = c_0 = 1$. The concatenation relations of Equation (6.6) can be seen to correspond to the substitution rule

$$A \mapsto ABC, \quad B \mapsto A\overline{B}\overline{C}, \quad C \mapsto A\overline{\overline{B}}\overline{\overline{C}},$$

which upon completion to a nine-letter substitution rule becomes

$$\begin{aligned} A &\mapsto ABC, & \overline{A} &\mapsto \overline{A}\overline{B}\overline{C}, & \overline{\overline{A}} &\mapsto \overline{\overline{A}}\overline{\overline{B}}\overline{\overline{C}}, \\ B &\mapsto A\overline{B}\overline{C}, & \overline{B} &\mapsto \overline{A}\overline{\overline{B}}\overline{C}, & \overline{\overline{B}} &\mapsto \overline{\overline{A}}\overline{B}\overline{C}, \\ C &\mapsto A\overline{\overline{B}}\overline{\overline{C}}, & \overline{\overline{C}} &\mapsto \overline{A}\overline{B}\overline{\overline{C}}, & \overline{\overline{\overline{C}}} &\mapsto \overline{\overline{\overline{A}}}\overline{\overline{B}}\overline{\overline{\overline{C}}}. \end{aligned} \quad (6.7)$$

Clearly, by induction, this rule gives rise to the concatenation relations

$$\begin{aligned} A_{k+1} &= A_k B_k C_k, \\ B_{k+1} &= A_k \overline{B_k} \overline{C_k}, \\ C_{k+1} &= A_k \overline{\overline{B_k}} \overline{\overline{C_k}} \end{aligned}$$

which has the same structure as Equation (6.6), but work on a nine-letter alphabet instead of a three-letter alphabet.

The fixed point is obtained by iteration on the initial letter A ,

$$\begin{aligned} A &\mapsto ABC \mapsto ABCA\overline{B}\overline{C}A\overline{\overline{B}}\overline{\overline{C}} \\ &\mapsto ABCA\overline{B}\overline{C}A\overline{\overline{B}}\overline{\overline{C}}ABCA\overline{B}\overline{C}A\overline{\overline{B}}\overline{\overline{C}}ABCA\overline{B}\overline{C}A\overline{\overline{B}}\overline{\overline{C}}ABCA\overline{B}\overline{C}A\overline{\overline{B}}\overline{\overline{C}} \mapsto \dots \end{aligned}$$

This fixed point is mapped to the ternary sequence of coefficients of $P_k(x)$ by the factor map

$$\varphi^{(3)}: \begin{cases} A, B, C \mapsto 1, \\ \overline{A}, \overline{B}, \overline{C} \mapsto \omega, \\ \overline{\overline{A}}, \overline{\overline{B}}, \overline{\overline{C}} \mapsto \omega^2. \end{cases} \quad (6.8)$$

Proposition 6.2. *The balanced weight sequence of the nine-letter substitution has Lebesgue spectrum.*

Proof. The second iterate of the letter A is $ABCA\overline{B}\overline{C}A\overline{\overline{B}}\overline{\overline{C}}$, which shows that the letter A can be preceded by \overline{B} or $\overline{\overline{B}}$. Hence the letter A has two distinct neighbourhoods, so, by Pansiot's lemma 2.12, the sequence is aperiodic.

The instruction matrices and the substitution matrix can be read off from the substitution rule of Equation (6.7) and are given by

$$R_0 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $R_0 + R_1 + R_2$. As $M^3 > 0$, the substitution is primitive. We find that the leading eigenvalue of the matrix M is 3 and $v_{\text{PF}} = \frac{1}{9}(1, 1, 1, 1, 1, 1, 1, 1, 1)$ for the corresponding Perron–Frobenius vector.

To determine the extremal points of the spectral hull \mathcal{K}^* , we compute the ergodic decomposition of the bi-substitution and obtain

$$\begin{aligned} E_1 &= \{A A, B B, C C, \overline{B} \overline{B}, \overline{C} \overline{C}, \overline{B} \overline{B}, \overline{C} \overline{C}, \overline{A} \overline{A}, \overline{A} \overline{A}\}, \\ E_2 &= \{A \overline{A}, B \overline{B}, C \overline{C}, \overline{B} \overline{B}, \overline{C} \overline{C}, \overline{B} \overline{B}, \overline{C} \overline{C}, \overline{A} \overline{A}, \overline{A} \overline{A}\}, \\ E_3 &= \{A \overline{A}, B \overline{B}, C \overline{C}, \overline{B} \overline{B}, \overline{C} \overline{C}, \overline{B} \overline{B}, \overline{C} \overline{C}, \overline{A} \overline{A}, \overline{A} \overline{A}\}, \end{aligned}$$

for the ergodic classes, and the transient class is formed by the rest of the 54 two letter words. We apply Proposition 3.5 to determine $v \in \mathcal{K}$. The extremal points \mathcal{K}^* are given by

$$(1, 1, 1), \left(1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), \left(1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),$$

and the $v \in \mathcal{K}^*$ are given by inserting the values of extremal points back into $v \in \mathcal{K}$.

As the PF vector is $\frac{1}{9}(1, 1, 1, 1, 1, 1, 1, 1)$, we have $\widehat{\Sigma}(0) = \frac{1}{9} \sum_{\gamma \in \mathcal{A}} e_{\gamma\gamma}$. Since this is a length 3 substitution, $q = 3$, so $\Delta_1(1) = \{2\}$. By applying Equation (3.4), we have

$$\begin{aligned} \widehat{\Sigma}(1) &= (3\mathbf{I} - R_2 \otimes R_0)^{-1} (R_0 \otimes R_1 + R_1 \otimes R_2) \widehat{\Sigma}(0) \\ &= \frac{1}{27} (0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1, \\ &\quad 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, \\ &\quad 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0). \end{aligned}$$

Using Theorem 3.3 for $k = 2$ and $p = 1$,

$$\begin{aligned} \widehat{\Sigma}(2) &= \frac{1}{3} \left((R_0 \otimes R_2 \widehat{\Sigma}(0) + (R_1 \otimes R_0 + R_2 \otimes R_1) \widehat{\Sigma}(1) \right) \\ &= \frac{1}{27} (0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, \\ &\quad 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \\ &\quad 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0). \end{aligned}$$

Using Theorem 3.3, we obtain

$$\begin{aligned}
\widehat{\Sigma}(3) &= \frac{1}{3} \sum_{j \in [0,3)} (R_j \otimes R_{j+3}) \widehat{\Sigma}(\lfloor j+3 \rfloor) \\
&= \frac{1}{3} (R_0 \otimes R_0 + R_1 \otimes R_1 + R_2 \otimes R_2) \widehat{\Sigma}(1) \\
&= \frac{1}{27} (1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, \\
&\quad 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, \\
&\quad 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1).
\end{aligned}$$

By direct computation, and Theorem 3.3, we check that $\widehat{\Sigma}(9) = \widehat{\Sigma}(6) = \widehat{\Sigma}(3)$, $\widehat{\Sigma}(10) = \widehat{\Sigma}(7) = \widehat{\Sigma}(1)$ and $\widehat{\Sigma}(11) = \widehat{\Sigma}(8) = \widehat{\Sigma}(2)$. We can then argue as shown before in the real weight case and obtain

$$\widehat{\Sigma}(3n+1) = \widehat{\Sigma}(1), \widehat{\Sigma}(3n+2) = \widehat{\Sigma}(2), \widehat{\Sigma}(3n) = \widehat{\Sigma}(3).$$

As before, $\lambda_{v_1} = \delta_0$, where the vector v_1 is given by the extreme point $(1, 1, 1)$, and using the Fourier coefficients and the identities mentioned above,

$$\widehat{\lambda}_{v_2}(3n) = 0, \widehat{\lambda}_{v_2}(3n+1) = 0 \text{ and } \widehat{\lambda}_{v_2}(3n+2) = 0,$$

and the same holds for v_3 , so both λ_{v_2} and λ_{v_3} are Lebesgue measure. Therefore the balance weight nine-letter substitution sequence has Lebesgue spectrum. \square

Aside from using Bartlett's algorithm to verify the existence of Lebesgue measure in the spectrum, we can also observe that the three and the nine-letter sequences are in fact MLD. It suffices to consider words of length four, many of which only have a single pre-image under $\varphi^{(3)}$ of Equation (6.8). An example is $111\omega^2$, for which the only pre-image is $ABC\overline{\overline{A}}$. Due to repetitivity, we can therefore determine the three sublattices locally, and hence locally derive the nine-letter sequence.

Finally, we consider one more example, which is the case when $n = 4$.

Example 6.2.2. We start by defining four sequences of polynomials,

$(P_k(x))_{k \in \mathbb{N}_0}$, $(Q_k(x))_{k \in \mathbb{N}_0}$, $(R_k(x))_{k \in \mathbb{N}_0}$ and $(S_k(x))_{k \in \mathbb{N}_0}$, where P_k , Q_k , R_k and S_k all have degree 4^k . They are determined by the initial choice

$P_0(x) = Q_0(x) = R_0(x) = S_0(x) = x$ together with the recurrence relations

$$\begin{pmatrix} P_{k+1}(x) \\ Q_{k+1}(x) \\ R_{k+1}(x) \\ S_{k+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x^{4^k} & x^{2 \cdot 4^k} & x^{3 \cdot 4^k} \\ 1 & ix^{4^k} & -x^{2 \cdot 4^k} & -ix^{3 \cdot 4^k} \\ 1 & -x^{4^k} & x^{2 \cdot 4^k} & -x^{3 \cdot 4^k} \\ 1 & -ix^{4^k} & -x^{2 \cdot 4^k} & +ix^{3 \cdot 4^k} \end{pmatrix} \begin{pmatrix} P_k(x) \\ Q_k(x) \\ R_k(x) \\ S_k(x) \end{pmatrix} \quad (6.9)$$

where $k \in \mathbb{N}_0$. By construction,

$$P_k(x) = \sum_{n=1}^{4^k} \varepsilon_n x^n,$$

where each coefficient ε_n is either 1 or -1 or i or $-i$, we can define a quaternary sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in \{1, -1, i, -i\}^{\mathbb{N}}$ from the coefficients. From here and (henceforth) we use the convention of a single bar to describe multiplication by i , double bar to describe multiplication by -1 and triple bar to describe multiplication by $-i$. The notation is in line with the fact $i^2 = -1$ and $i^3 = -i$. If $a_k = \varepsilon_1 \cdots \varepsilon_{4^k} \in \{1, -1, i, -i\}$ denotes the word of length 4^k of coefficients $P_k(x)$, b_k denotes the corresponding word for $Q_k(x)$, $c_k(x)$ denotes the corresponding word for $R_k(x)$ and $d_k(x)$ denotes the corresponding word for $S_k(x)$, then the recurrence relations of Equation (6.9) correspond to the concatenation relations

$$\begin{aligned} a_{k+1} &= a_k b_k c_k d_k, \\ b_{k+1} &= a_k \overline{b_k} \overline{\overline{c_k}} \overline{\overline{\overline{d_k}}}, \\ c_{k+1} &= a_k \overline{\overline{b_k}} \overline{\overline{c_k}} \overline{\overline{\overline{d_k}}}, \\ d_{k+1} &= a_k \overline{\overline{\overline{b_k}}} \overline{\overline{\overline{c_k}}} \overline{\overline{\overline{\overline{d_k}}}} \end{aligned} \quad (6.10)$$

on words in the four letter alphabet $\{1, -1, i, -i\}$, with initial values $a_0 = b_0 = c_0 = d_0 = 1$. The concatenation relations of Equation (6.10) can be seen to correspond to the substitution rule

$$A \mapsto ABCD, \quad B \mapsto A\overline{B}\overline{\overline{C}}\overline{\overline{\overline{D}}}, \quad C \mapsto A\overline{\overline{B}}C\overline{\overline{D}}, \quad D \mapsto A\overline{\overline{\overline{B}}}\overline{\overline{\overline{C}}}\overline{\overline{\overline{\overline{D}}}},$$

which up to completion to a 16-letter substitution rule becomes

$$\begin{aligned} A &\mapsto ABCD, & \overline{A} &\mapsto \overline{A}\overline{B}\overline{C}\overline{D}, & \overline{\overline{A}} &\mapsto \overline{\overline{A}}\overline{\overline{B}}\overline{\overline{C}}\overline{\overline{D}}, & \overline{\overline{\overline{A}}} &\mapsto \overline{\overline{\overline{A}}}\overline{\overline{\overline{B}}}\overline{\overline{\overline{C}}}\overline{\overline{\overline{\overline{D}}}}, \\ B &\mapsto A\overline{B}\overline{\overline{C}}\overline{\overline{\overline{D}}}, & \overline{B} &\mapsto \overline{A}\overline{\overline{B}}\overline{\overline{\overline{C}}}D, & \overline{\overline{B}} &\mapsto \overline{\overline{A}}\overline{\overline{\overline{B}}}C\overline{D}, & \overline{\overline{\overline{B}}} &\mapsto \overline{\overline{\overline{A}}}\overline{\overline{\overline{\overline{B}}}}\overline{\overline{C}}\overline{\overline{\overline{\overline{D}}}}, \\ C &\mapsto A\overline{\overline{B}}C\overline{\overline{D}}, & \overline{\overline{C}} &\mapsto \overline{A}\overline{\overline{\overline{B}}}\overline{\overline{\overline{\overline{C}}}}\overline{\overline{\overline{\overline{D}}}}, & \overline{\overline{\overline{C}}} &\mapsto \overline{\overline{A}}\overline{\overline{\overline{\overline{B}}}}\overline{\overline{\overline{\overline{\overline{C}}}}}D, & \overline{\overline{\overline{\overline{C}}}} &\mapsto \overline{\overline{\overline{\overline{A}}}}\overline{\overline{\overline{\overline{\overline{B}}}}}\overline{\overline{\overline{\overline{\overline{\overline{C}}}}}}\overline{\overline{\overline{\overline{\overline{\overline{D}}}}}}, \\ D &\mapsto A\overline{\overline{\overline{B}}}\overline{\overline{\overline{\overline{C}}}}\overline{\overline{\overline{\overline{\overline{D}}}}}, & \overline{\overline{\overline{D}}} &\mapsto \overline{A}\overline{\overline{\overline{B}}}\overline{\overline{\overline{\overline{\overline{C}}}}}\overline{\overline{\overline{\overline{\overline{\overline{D}}}}}}, & \overline{\overline{\overline{\overline{D}}}} &\mapsto \overline{\overline{A}}\overline{\overline{\overline{\overline{B}}}}\overline{\overline{\overline{\overline{\overline{\overline{C}}}}}}\overline{\overline{\overline{\overline{\overline{\overline{\overline{D}}}}}}}, \end{aligned}$$

Clearly, by induction, this rule gives rise to the concatenation relations

$$\begin{aligned} A_{k+1} &= A_k B_k C_k D_k, \\ B_{k+1} &= A_k \overline{B_k} \overline{\overline{C_k}} \overline{\overline{\overline{D_k}}}, \\ C_{k+1} &= A_k \overline{\overline{B_k}} \overline{C_k} \overline{\overline{D_k}}, \\ D_{k+1} &= A_k \overline{\overline{\overline{B_k}}} \overline{\overline{\overline{C_k}}} \overline{\overline{\overline{D_k}}}, \end{aligned}$$

which has the same structure as Equation (6.10), but work on the 16-letter alphabet instead of a four-letter alphabet.

The fixed point is obtained by iteration on the initial letter A ,

$$A \mapsto A B C D \mapsto A B C D A \overline{B} \overline{\overline{C}} \overline{\overline{\overline{D}}} A \overline{\overline{B}} \overline{\overline{\overline{C}}} \overline{\overline{\overline{\overline{D}}}} A \overline{\overline{\overline{\overline{B}}}} \overline{\overline{\overline{\overline{\overline{C}}}}} \overline{\overline{\overline{\overline{\overline{D}}}}} \mapsto \dots$$

This fixed point is mapped to the quaternary sequence of coefficients of $P_k(x)$ by the factor map

$$\varphi^{(4)}: \begin{cases} A, B, C, D \mapsto 1, \\ \overline{A}, \overline{B}, \overline{C}, \overline{D} \mapsto i, \\ \overline{\overline{A}}, \overline{\overline{B}}, \overline{\overline{C}}, \overline{\overline{D}} \mapsto -1, \\ \overline{\overline{\overline{A}}}, \overline{\overline{\overline{B}}}, \overline{\overline{\overline{C}}}, \overline{\overline{\overline{D}}} \mapsto -i. \end{cases}$$

As before, the four-letter and 16-letter sequences are mutually locally derivable, and again it is possible to find words of length 4 that only have a single ancestor under $\varphi^{(4)}$. One example is $111 - 1$ whose ancestor is $BCD\overline{\overline{A}}$.

In the same way, starting from the $n \times n$ Fourier matrix, we can construct substitution rules for any $n > 1$, which all have absolute continuous components in their spectra. The general structure is clear from the examples given above. The substitutions act on n^2 letters with n ‘basic’ letters that appear in n different ‘flavours’, each distinguished by the number of bars, from 0 to $n - 1$. The distribution of bars in the image of the four basic letters can be read off directly from the Fourier matrix, and the remainder of the substitution is then fixed by cyclic symmetry under the bar operation. The corresponding factor map $\varphi^{(n)}$ identifies all basic letters, and the image only depends on the number of bars, giving the corresponding power of $\exp(2\pi i/n)$.

7

CONCLUSION

Contents

7.1	Summary of results obtained	86
7.2	Future questions	87

7.1 Summary of results obtained

In this thesis, we established a connection between the root- N property in Equation (5.1) and the property that any sequence that satisfies it has absolutely continuous/Lebesgue spectrum. In doing so and applying Bartlett's algorithm [10, 11], we were able to answer a question raised by Yee, Lafrance and Rampersad [34] regarding whether the Rudin–Shapiro-like sequence satisfies the root- N property. This result reveals that although the Rudin–Shapiro-like sequence shares similarities with the Rudin–Shapiro sequence, in terms of spectral structure, they are actually rather different. Note that this root- N property result regarding the Rudin–Shapiro-like sequence was independently obtained by Allouche [1] by using a different method. His result relies on exploiting the recursion relations of the sequence [34, Prop. 1].

Once we established the connection between the root- N property and the spectral properties of a sequence, we revisited Rudin's original construction [44] of the binary Rudin–Shapiro sequence and generalized it. Our extension yielded more balanced weight sequences with both real and complex coefficients that satisfy the root- N property. We then presented a general construction on how to derive aperiodic, constant length substitution sequences from this extension. For the real coefficients case, we proved that the balanced weight, binary sequences and the corresponding substitution sequences from our extension are mutually locally derivable, hence there is a topological conjugacy between them. It then follows from [49] that the spectral properties of the substitution sequences are the same as the binary, balanced weight sequences. For the complex coefficients case, we constructed the first known example of aperiodic, constant length substitution of length three with absolutely continuous/Lebesgue spectrum and verified its spectral properties using Bartlett's algorithm. Using the notion of mutual local derivability, we showed that the ternary balanced weight sequences and quaternary balanced weight sequences are mutually locally derivable to their corresponding substitution sequences. This gives us an alternative way of showing that the substitution sequences have absolutely continuous/Lebesgue spectrum without using Bartlett's algorithm.

7.2 Future questions

The first question concerns Bartlett's algorithm in Chapter 3. This algorithm currently only allows us to compute *aperiodic, constant length substitutions on \mathbb{Z}^d* and it relies heavily on the lattice structure behind the substitution, which allows us to exploit the arithmetic structure mentioned in Section 3.1. It leads us to ask the following question:

Question 7.2.1. Is there an analogue of Bartlett's results in the non-constant length setting?

It is not really clear whether it is possible to extend Bartlett's results to the non-constant length situation as letters are mapped into words of different length. The arithmetic behind the substitutions is much more complicated. One would need a completely different notion to capture the arithmetic behind the substitution than q -adic substitution in Section 3.1.

For our second question, we recall in Chapter 5, we showed in Proposition 5.12 that the hull of the ± 1 sequences obtained from the root- N construction and the hull of the corresponding substitutions are mutually locally derivable. In Chapter 6, we showed that the hull of the ternary sequence and the hull of the corresponding length 3 substitution are mutually locally derivable, and this holds also for the hull of the quaternary sequence and the hull of the length 4 substitution. A natural question to ask is the following:

Question 7.2.2. Can we show in general that the hull of the complex weights sequences from the root- N construction and the hull of the corresponding constant length substitutions are mutually locally derivable?

We do not know how difficult this question would turn out to be. In Proposition 5.12, we identified a word that has a unique pre-image and this word is legal for the particular construction (S_{σ_k}) because of the composition property. In the complex coefficients case, we do not have this composition property, hence it will be more difficult.

For our third question, we recall in Chapter 2, we introduced the notion of sliding block codes and, by Theorem 2.15, we know that topological conjugacy

between two systems is given by a sliding block code. By a theorem of Coven, Dykstra, Keane and LeMasurier [21, Thm. 3], if there is a conjugacy between two substitution systems, the sliding block code will be of maximum radius 3. So to determine whether there is a topological conjugacy between two systems, one might be tempted to perform a brute force search for such a sliding block code. Unfortunately, this becomes a very inefficient method when the size of the alphabet is large. Recently, Coven, Dekking and Keane [20] gave an algorithm to produce a list of injective substitutions of the same length that generate topologically conjugate systems. The idea behind such an approach is to connect substitutions and graph homomorphism [20, Sect. 7]. The authors produced a list of systems that are conjugate to the Thue–Morse substitution [20, Sect. 10] by considering 406 cases of graph homomorphisms. Motivated by the studies of absolutely continuous/Lebesgue spectrum of substitution systems, we are interested in the following question:

Question 7.2.3. Would we be able to use the same approach to produce a list of systems of the same length that are conjugate to the Rudin–Shapiro sequence?

We believe the answer to this question is yes, based on the evidence from [20, Procedure 8.1]. We start by considering all the length 3 words that are admitted by our substitution S_+ of Equation (5.5); these can be computed easily by using the program of Balchin and Rust [9]. The set of length 3 words is $\{ABA, AB\bar{A}, A\bar{B}A, A\bar{B}\bar{A}, BAB, B\bar{A}B, B\bar{A}\bar{B}, B\bar{A}\bar{B}, \bar{B}AB, \bar{B}A\bar{B}, \bar{B}A\bar{B}, \bar{B}A\bar{B}, \bar{A}B\bar{B}, \bar{A}B\bar{A}, \bar{A}\bar{B}A, \bar{A}\bar{B}\bar{A}\}$. Using such an ordering, these length 3 words are then mapped into the letters in the new alphabet

$$\mathcal{A}^{[3]} := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}.$$

One can then form graphs G_1 , $G_{2,0}$ and $G_{2,1}$; see [20, Sect. 7] for notations and background on substitutions and graphs. We then try to determine all the graph epimorphisms from G_1 to $G_{2,0}$ and $G_{2,1}$. Given the size of new alphabet $\mathcal{A}^{[3]}$, it is likely we would need computer program to aid the calculation. Once we determine all the graph epimorphisms, this would give us a list of all possible substitutions that are conjugate to S_+ , after discarding the ones that are not primitive, in accordance with [20, Procedure 8.1].

For our fourth question, we recall in Chapter 5 and Chapter 6, we see that under the map φ , the letters in the substitution sequences are assigned to the values in the binary, ternary and quaternary versions of the sequences. Such assignments are natural as the substitution sequences are derived according to the appropriate labelling in our modification of Rudin's construction. One natural question to ask is that

Question 7.2.4. Would the spectrum change if we assign different underlying numbers to the letters in the substitution sequences?

We believe the answer to the question is yes. One way to prove absolute continuity in the spectrum is via the autocorrelation coefficients argument [6, Sect. 10.2]. The authors proved that the autocorrelation coefficients vanish except at 0, which gives rise to an autocorrelation measure δ_0 , the Dirac measure at 0. The Fourier transform of the Dirac measure at 0 is the Lebesgue measure, hence establishing absolute continuity. If we assign the underlying numbers to the letters in the substitution sequences randomly, then the autocorrelation coefficients might not necessarily vanish at all positions, which means the spectrum might not necessarily be absolutely continuous.

For our fourth question, we recall in Chapter 5, the infinite sequences obtained in the manner of Theorem 5.10 still gives convergence in the local topology, due to our underlying construction. However, for the substitution sequences obtained, it is no longer a fixed point of a primitive substitution of finite length, so we do not know much about the corresponding hull. So it leads us to ask the following question:

Question 7.2.5. Is there a framework in symbolic dynamical systems that allows us to get some information or understand better of the hull of the substitutions $(S_{\sigma_k})_{k \in \mathbb{N}_0}$?

According to the terminology of Ferenczi [27], the infinite sequence obtained in the manner of Theorem 5.10 is called an *S-adic expansion* and the sequence of substitutions $(S_{\sigma_k})_{k \in \mathbb{N}_0}$ are referred to as *directive sequence*. Using such terminology, we can translate our studies of $(S_{\sigma_k})_{k \in \mathbb{N}_0}$ into the frame work of *S-adic substitutions* [13, 14], where much is understood about minimality and ergodicity of *S-adic systems* [13, Sect. 5].

For our final question, we recall in chapter 5, we stated Frank's construction/characterization of substitution systems in \mathbb{Z}^d having absolutely continuous spectrum. In Chapter 5 and Chapter 6, we provided an alternative construction/characterization of substitution systems that have absolutely continuous spectrum, without satisfying the last condition of Theorem 5.1. This leads us to ask the following question:

Question 7.2.6. Does there exist a simple, general criterion to tell whether a substitution-based sequence has any absolutely continuous spectrum?

We do not know how hard this question would turn out to be. One natural starting point would be to use the *unraveling* technique Frank mentioned [29, Sect. 5], and unravel the higher dimensional substitution systems of her construction into one dimension. Since her substitutions also employ the barred and unbarred versions of the letters to be $+1$ and -1 respectively, we can then obtain the binary versions of these substitution sequences and verify that they satisfy the root- N property.

BIBLIOGRAPHY

- [1] J.-P. Allouche. On a Golay-Shapiro-like sequence. *Unif. Distrib. Theory*, 11(2):205–210, 2016.
- [2] J.-P. Allouche and P. Liardet. Generalized Rudin-Shapiro sequences. *Acta Arith.*, 60(1):1–27, 1991.
- [3] J.-P. Allouche and J. Shallit. *Automatic sequences: Theory, applications, generalizations*. Cambridge University Press, Cambridge, 2003.
- [4] M. Baake, D. Damanik, and U. Grimm. What is ... aperiodic order? *Notices Amer. Math. Soc.*, 63(6):647–650, 2016.
- [5] M. Baake and F. Gähler. Pair correlations of aperiodic inflation rules via renormalisation: some interesting examples. *Topology Appl.*, 205:4–27, 2016.
- [6] M. Baake and U. Grimm. *Aperiodic order. Vol. 1*, volume 149 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2013.
- [7] M. Baake and D. Lenz. Spectral notions of aperiodic order. *Discrete Contin. Dyn. Syst. Ser. S*, 10(2):161–190, 2017.
- [8] M. Baake, M. Schlottmann, and P. D. Jarvis. Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability. *J. Phys. A*, 24(19):4637–4654, 1991.
- [9] S. Balchin and D. Rust. Computations for symbolic substitutions. *J. Integer Seq.*, 20(4), 2017.
- [10] A. Bartlett. Spectral theory of \mathbb{Z}^d substitutions. *Ergod. Th. & Dynam. Syst.*, in press, Preprint *arXiv:1410.8106*, 2014.
- [11] A. Bartlett. *Spectral Theory of \mathbb{Z}^d Substitutions*. ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)—University of Washington.
- [12] L. Bendersky. Quasicrystal with one-dimensional translational symmetry and a tenfold rotation axis. *Phys. Rev. Lett.*, 55:1461–1463, 1985.

- [13] V. Berthé and V. Delecroix. Beyond substitutive dynamical systems: S -adic expansions. In *Numeration and substitution 2012*, RIMS Kôkyûroku Bessatsu, B46, pages 81–123. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.
- [14] V. Berthé, W. Steiner, and J. Thuswaldner. Geometry, dynamics, and arithmetic of s -adic shifts. *Preprint arXiv:1410.0331*, 2014.
- [15] J. Brillhart, P. Erdős, and P. Morton. On sums of Rudin-Shapiro coefficients. II. *Pacific J. Math.*, 107(1):39–69, 1983.
- [16] L. Chan and U. Grimm. Spectrum of a Rudin-Shapiro-like sequence. *Adv. in Appl. Math.*, 87:16–23, 2017.
- [17] L. Chan and U. Grimm. Substitution-based sequences with absolutely continuous diffraction. *Journal of Physics: Conference series*, 809(012027), 2017.
- [18] J. Coquet. A summation formula related to the binary digits. *Invent. Math.*, 73(1):107–115, 1983.
- [19] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskii.
- [20] E. M. Coven, F. M. Dekking, and M. S. Keane. Topological conjugacy of constant length substitution dynamical systems. *Indag. Math. (N.S.)*, 28(1):91–107, 2017.
- [21] E. M. Coven, A. Dykstra, M. Keane, and M. LeMasurier. Topological conjugacy to given constant length substitution minimal systems. *Indag. Math. (N.S.)*, 25(4):646–651, 2014.
- [22] N. G. de Bruijn. Algebraic theory of Penrose’s nonperiodic tilings of the plane. I, II. *Nederl. Akad. Wetensch. Indag. Math.*, 43(1):39–52, 53–66, 1981.
- [23] F. M. Dekking. The spectrum of dynamical systems arising from substitutions of constant length. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 41(3):221–239, 1978.

- [24] J. M. Dumont and A. Thomas. Systemes de numeration et fonctions fractales relatifs aux substitutions. *Theoret. Comput. Sci.*, 65(2):153–169, 1989.
- [25] F. Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergod. Th. & Dynam. Syst.*, 20(4):1061–1078, 2000.
- [26] S. Dworkin. Spectral theory and x-ray diffraction. *J. Math. Phys.*, 34(7):2965–2967, 1993.
- [27] S. Ferenczi. Rank and symbolic complexity. *Ergod. Th. & Dynam. Syst.*, 16(4):663–682, 1996.
- [28] N. P. Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.
- [29] N. P. Frank. Substitution sequences in \mathbb{Z}^d with a non-simple Lebesgue component in the spectrum. *Ergod. Th. & Dynam. Syst.*, 23(2):519–532, 2003.
- [30] M. Gardner. Extraordinary non-periodic tiling that enriches the theory of tiles. *Sci. Amer.*, 236:110–121, 1977.
- [31] M. J. E Golay. Statistic multislit spectrometry and its application to the panoramic display of infrared spectra. *J. Optical Soc. America*, 41:468–472, 1951.
- [32] A. Hof. On diffraction by aperiodic structures. *Comm. Math. Phys.*, 169(1):25–43, 1995.
- [33] T. Ishimasa, H.-U Nissen, and Y. Fukano. New ordered state between crystalline and amorphous in ni-cr particles. *Phys. Rev. Lett.*, 55:511–513, 1985.
- [34] P. Lafrance, N. Rampersad, and R. Yee. Some properties of a Rudin-Shapiro-like sequence. *Adv. in Appl. Math.*, 63:19–40, 2015.
- [35] D. Levine and P. J. Steinhardt. Quasicrystals: A new class of ordered structures. *Phys. Rev. Lett.*, 53:2477–2480, 1984.

- [36] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.
- [37] B. Mossé. Reconnaissabilité des substitutions et complexité des suites automatiques. *Bull. Soc. Math. France*, 124(2):329–346, 1996.
- [38] D. J. Newman. On the number of binary digits in a multiple of three. *Proc. Amer. Math. Soc.*, 21:719–721, 1969.
- [39] J. Pansiot. Decidability of periodicity for infinite words. *RAIRO Inform. Théor. Appl.*, 20(1):43–46, 1986.
- [40] L. R. Penrose and R. Penrose. Puzzles for christmas. *New Scientist*, pages 1580–1581, 1597–1598, 1958.
- [41] R. Penrose. The role of aesthetics in pure and applied mathematical research. *Bull. Inst. Math. Appl.*, 10:266–271, 1974.
- [42] M. Queffélec. *Substitution dynamical systems—spectral analysis*, volume 1294 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2010.
- [43] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.
- [44] W. Rudin. Some theorems on Fourier coefficients. *Proc. Amer. Math. Soc.*, 10:855–859, 1959.
- [45] H. S. Shapiro. *Extremal problems for polynomials and power series*. ProQuest LLC, Ann Arbor, MI, 1953. Thesis (Ph.D.)—Massachusetts Institute of Technology.
- [46] D. Shechtman, I. Blech, D. Gratias, and Cahn. J. W. Metallic phase with long-range orientational order and no translation symmetry. *Phys. Rev. Lett.*, 53:1951–1953, 1984.
- [47] B. Solomyak. Nonperiodicity implies unique composition for self-similar translationally finite tilings. *Discrete Comput. Geom.*, 20(2):265–279, 1998.

-
- [48] W. Tadej and K. Życzkowski. A concise guide to complex Hadamard matrices. *Open Syst. Inf. Dyn.*, 13(2):133–177, 2006.
- [49] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.
- [50] M. Wang, H. Chen, and K. H. Kuo. Two-dimensional quasicrystal with eightfold rotational symmetry. *Phys. Rev. Lett.*, 59:1010–1013, 1987.
- [51] H. Wielandt. Unzerlegbare, nicht negative Matrizen. *Math. Z.*, 52:642–648, 1950.